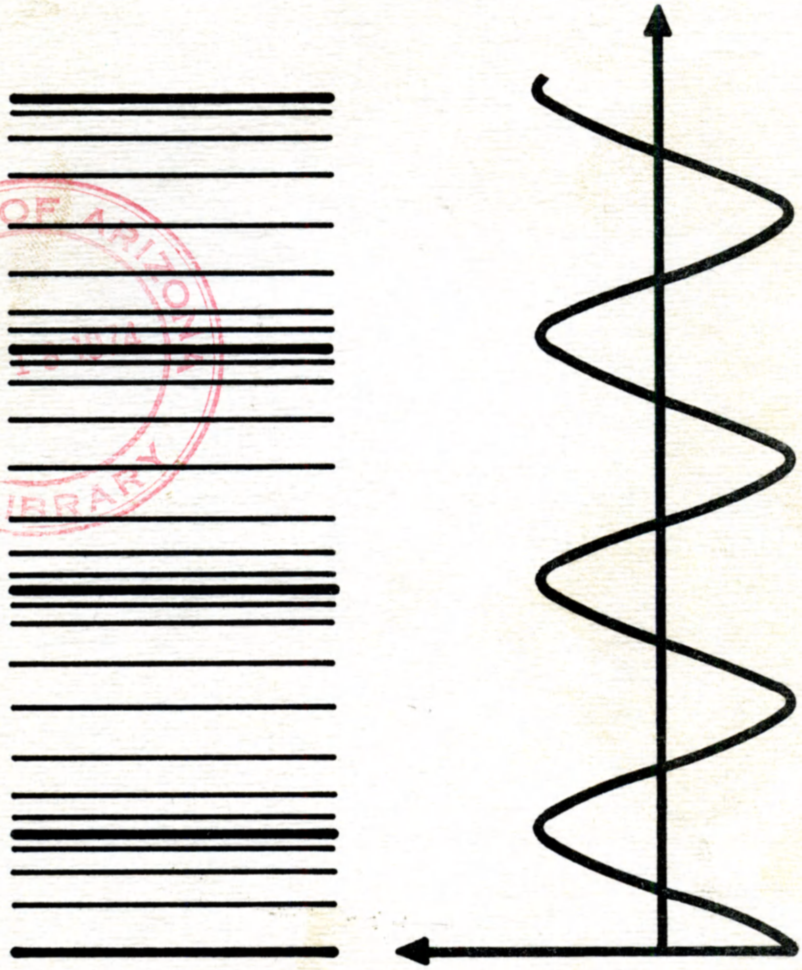
OPTICAL SCIENCES CENTER

TECHNICAL REPORT 83

JUNE 1974



Diffraction of Light by Sound Waves

Virendra N. Mahajan

ERRATA

Technical Report 83, "Diffraction of Light by Sound Waves"

Page 4: The line above Equation (7) should read:

Using the identity (ref. 7, p. 22)

Page 5: Equation (8) should read:

$$E_e = \tfrac{1}{2} \sum_{\ell=-\infty}^{\infty} J_\ell(\Delta\phi)\, e^{i[(\omega+\ell\Omega)t-kz-\ell Kx]} + \text{c.c.},$$

Page 24: A plus sign should be inserted in the first line of Equation (53) so as to read:

$$\tfrac{1}{2} \sum_{\ell=-\infty}^{\infty} \left[\frac{d^2U_{\ell x}}{dz^2}\,\hat{x} + \frac{d^2U_{\ell y}}{dz^2}\,\hat{y} - k_\ell^{\,2}\vec{U} + i\left(\vec{k}_\ell \frac{dU_{\ell z}}{dz} - 2k_{\ell z}\frac{d\vec{U}_\ell}{dz}\right)\right]$$

$$\cdot\, e^{i(\omega_\ell t - \vec{k}_\ell \cdot \vec{r})} + \text{c.c.}$$

$$= \frac{1}{v^2}\,\frac{1}{2} \sum_{\ell=-\infty}^{\infty} (i\omega_\ell)^2 \left[\vec{U}_\ell + \frac{\Delta n}{in}\left(\vec{U}_{\ell-1}e^{i\Phi} - \vec{U}_{\ell+1}e^{-i\Phi}\right)\right]$$

$$\cdot\, e^{i(\omega_\ell t - \vec{k}_\ell \cdot \vec{r})} + \text{c.c.} \qquad (53)$$

In Equation (54) the plus sign should be changed to a minus and the minus sign after the equal sign should be deleted.

$$\vec{k}_\ell \frac{dU_{\ell z}}{dz} - 2k_{\ell z}\frac{d\vec{U}_\ell}{dz} - \frac{\Delta n}{n}\frac{\omega_\ell^{\,2}}{v^2}\left(\vec{U}_{\ell-1}e^{i\Phi} - \vec{U}_{\ell+1}e^{-i\Phi}\right)$$

$$= i\left(\frac{\omega_\ell^{\,2}}{v^2} - k_\ell^{\,2}\right)\vec{U}_\ell. \qquad (54)$$

Page 28: The line above Equation (71) should read:

Using the identity for Bessel functions of the first kind (ref. 7, p. 17), ERRATA (continued)

Page 31: The first term of Equation (80) should read:

$$\frac{dV_0}{dz} - \frac{\zeta}{2L} V_1 = 0 \tag{80}$$

The first term of Equation (81) should read:

$$\frac{dV_1}{dz} + \frac{\zeta}{2L} V_0 = i \frac{Q}{2L} (1 - 2\alpha)V_1, \tag{81}$$

Page 49: $\vec{k}$ should be deleted from below the horizontal z axis.

Page 50: Reference 5 should read:

C. V. Raman and N. S. Nagendra Nath, "The diffraction of light by high frequency sound waves: Part I," Proc. Ind. Acad. Sci. 2A:406-412, 1935; "Part II," 2A:413-420, 1935; "Part III--Doppler effect and coherence phenomena," 3A:75-84, 1936; "Part IV--Generalised theory," 3A:119-125, 1936; "Part V--General considerations--oblique incidence and amplitude changes," 3A:459-465, 1936.

ADDENDA

Some of the work reported here has been published as follows:

V. N. Mahajan and J. D. Gaskill, "Diffraction of light by sound waves according to the vector wave equation," J. Opt. Soc. Am. 64, 400, 1974.

V. N. Mahajan and J. D. Gaskill, "Doppler interpretation of the frequency shifts of light diffracted by sound waves," J. Appl. Phys. 45, 2799, 1974.

V. N. Mahajan and J. D. Gaskill, "Bragg diffraction of light by a standing sound wave," to be published in the November 1974 issue of Optica Acta.

DIFFRACTION OF LIGHT BY SOUND WAVES

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Technical Report 83, June 1974

FOREWORD

This technical report is adapted from work submitted in partial fulfillment of requirements for the PhD preliminary examination in Optical Sciences.

ABSTRACT

Diffraction of light by a sinusoidal sound wave is discussed in detail. Assuming that the sound column modulates only the phase of the incident light in both time and space, the frequencies, wavevectors, and intensities of the diffracted waves are obtained for normal incidence. A transition length (width of sound beam) is defined, above which all diffraction effects disappear due to destructive interference. Constructive interference is obtained, however, provided the light is incident at the Bragg angle, in which case the diffracted beam appears to be reflected from the acoustic wavefronts. The transition length thus separates the region of multiple-order (Raman-Nath) diffraction from the region of single-order (Bragg) diffraction. It is found to be directly proportional to the square of the acoustic wavelength and inversely proportional to the optical wavelength.

In the case of Bragg diffraction, the energy is exchanged sinusoidally between the diffracted and undiffracted beams. Owing to the finite width of the sound beam, the Bragg condition is relaxed, and the effect can be used to control the direction and phase of the diffracted beam or to determine the angular distribution of the acoustic power.

Next, a particle picture of diffraction in terms of photons and phonons is given. The diffraction process is described as a single as well as a multiple three-particle interaction. The effects of finite optical and acoustic beamwidths and variation of acoustic frequency are considered in terms of momentum conservation.

Finally, an analysis based on Maxwell's equations for an arbitrarily polarized light beam propagating in an arbitrary direction is given using the partial-wave approach. A set of coupled difference-differential equations for the diffracted amplitudes is derived from the optical wave equation and analytic solutions are obtained in the Raman-Nath and Bragg regions of diffraction. The results for normal and Bragg incidence are obtained as special cases. Limits of the two regions are defined, thus giving a transition region in which numerical solutions can be obtained.

CONTENTS

1. INTRODUCTION

When a lighted object is observed through a transparent material in which a high-frequency sound wave is traveling, a set of closely spaced images is found. If the sound wave is made more intense, the intensity of each image varies, and their number increases. This is the phenomenon of diffraction of light by sound waves predicted by Brillouin[1] in 1922 and observed experimentally about a decade later by Debye and Sears,[2] Lucas and Biquard,[3] and Bär.[4]

A sound wave produces a periodic strain in the medium in which it is traveling, and this strain in turn produces variations of the refractive index of the medium; the index change is related to the strain by the photoelastic constant of the medium (Appendix A). Consequently, a light beam incident on the medium is perturbed as it traverses the sound column. When the length of interaction (i.e., the width of the sound beam) is small, the incident light is phase modulated in both space and time,[5] and the light is said to be Raman-Nath diffracted. The space modulation gives rise to a discrete set of propagation vectors distributed about the incident propagation vector, and the temporal modulation to the frequency shifts. Each propagation vector defines a diffraction order or, equivalently, an image. The frequency of the light in a particular order differs from that of the incident light by an integral multiple of the acoustic-wave frequency, the integer being the order number. This difference in frequency represents a Doppler shift due to the motion of the acoustic wavefronts (Appendix B).

All diffraction effects tend to disappear owing to destructive interference when the interaction length becomes sufficiently large. However, light can still be diffracted provided it is incident at an angle approximately equal to half the angle between the first and zero diffraction orders, but only one of the two first orders, besides the zero order, is observed.[6] The angle of incidence for maximum diffraction is called the Bragg angle, and the light is said to be Bragg diffracted. Since the Bragg angle depends on the acoustic frequency, the direction of the diffracted beam can be controlled by varying the frequency. Optimum diffraction can be maintained by steering the acoustic beam (Appendix C). Moreover, the phase of the diffracted wave depends on the phase of the acoustic wave. The application of Bragg diffraction to real-time correction of wavefronts distorted by atmospheric turbulence will be considered in a later report.

The dominant features of the diffraction process are discussed in Sections 2 and 3. Diffraction under a general set of conditions (e.g., light of arbitrary polarization and direction of propagation) is described in Section 4 where the optical wave equation is solved by assuming that the optical field inside the sound column is composed of a series of plane waves. Rigorous theory of this section confirms the conclusions of earlier sections. Interesting results are obtained in the case of diffraction from a standing sound wave and are discussed in Appendix D.

2. RAMAN-NATH AND BRAGG DIFFRACTION

Consider a unit-amplitude plane wave of light, of frequency ω, incident horizontally (z axis) on a transparent material of refractive index n. A sound wave of frequency Ω travels vertically (x axis) in this material with a velocity V. The sound wave produces a periodic strain in the medium, which in turn produces variations in its refractive index by means of the photoelastic effect. As shown in Fig. 1, longitudinal sound waves are accompanied by density variations, the index of refraction being higher in the compressed regions and lower in the rarefied regions. Transverse sound waves produce index variations without producing any density variations. In either case, the sound wave generates an index wave that may be represented by

$$\Delta n(x,t) = \Delta n \sin(\Omega t - Kx) \qquad \textit{Index Wave} \qquad (1)$$

where the acoustic wavenumber K, wavelength Λ, and frequency Ω are related according to

$$K = 2\pi/\Lambda = \Omega/V. \qquad (2)$$

The relationship between Δn and the acoustic power is discussed in Appendix A. The incident light field can be written as

$$E_i = \tfrac{1}{2} e^{i(\omega t - kz)} + \text{c.c.} \qquad (3)$$

where c.c. denotes complex conjugate, and k is the wavenumber of light in the medium and is related to its wavelength λ and frequency ω according to

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} n, \qquad (4)$$

where c is the velocity of light in vacuum. In terms of its wavelength λ_a in air (vacuum), $k = (2\pi/\lambda_a)n$. The light emerges from the sound column phase modulated; the magnitude of the phase modulation (often called the modulation index) is given by

$$\Delta\phi = kL \frac{\Delta n}{n} \qquad (5)$$

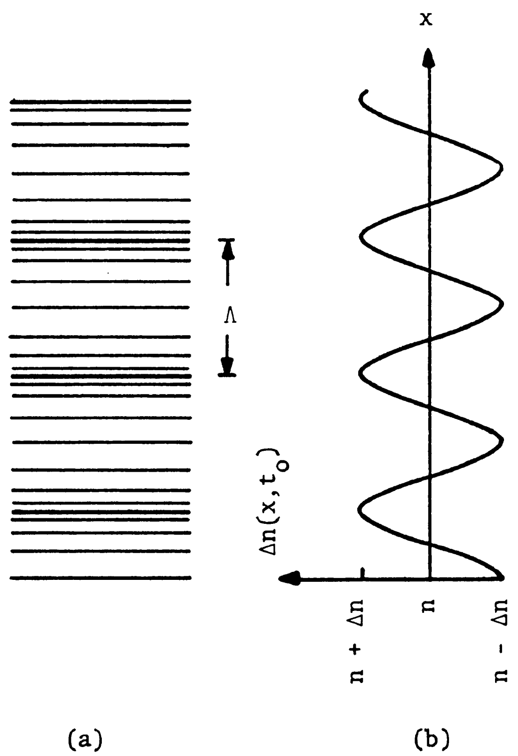


Fig. 1. Low-frequency longitudinal sound wave in a transparent material of refractive index n. *(a) Alternating regions of compression and rarefaction that travel at the sound velocity* V. *(b) Instantaneous spatial variation of the refractive index produced by the sound wave.*

where L is the length of the sound beam that the light traverses. (L is the width of the sound column and is called the interaction (or scattering) length.) The emergent light can therefore be written as

$$E_e = \tfrac{1}{2} e^{i(\omega t - kz)} e^{i\Delta\phi \sin(\Omega t - Kx)} + \text{c.c.} \qquad (6)$$

Using the identity (ref. 7, p. 17)

$$e^{ia \sin b} = \sum_{\ell=-\infty}^{\infty} J_\ell(a)\, e^{i\ell b} \qquad (7)$$

where J_ℓ is the ℓth-order Bessel function of the first kind, we can write Eq. (6) as

$$E_e = \frac{1}{2} \sum_{\ell=-\infty}^{\infty} J_\ell(\Delta\phi)\ e^{i[(\omega+\ell\Omega)t-kz-\ell Kz]} + \text{c.c.}, \tag{8}$$

or,

$$E_e = \frac{1}{2} \sum_{\ell=-\infty}^{\infty} J_\ell(\Delta\phi) e^{i(\omega_\ell t-\vec{k}_\ell\cdot\vec{r})} + \text{c.c.}, \tag{9}$$

where

$$\omega_\ell = \omega + \ell\Omega \tag{10}$$

$$\vec{k}_\ell = (\ell K,\ 0,\ k). \tag{11}$$

Thus the sound column diffracts the incident light into various orders ℓ as shown in Fig. 2. The frequency and wavevector of the light in the ℓth diffraction order are given by Eqs. (10) and (11), respectively.

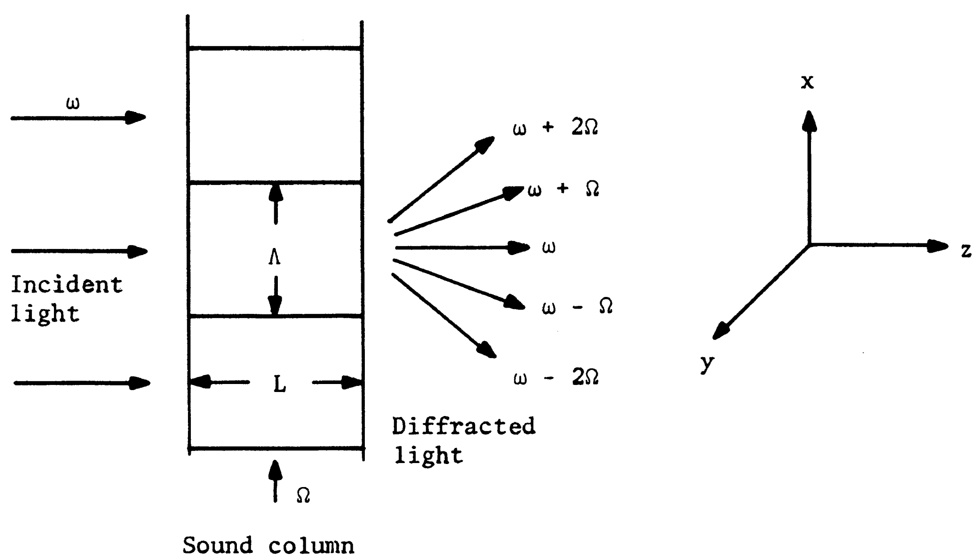


Fig. 2. Raman-Nath diffraction. Horizontal lines in sound column show equivalent acoustic wavefronts.

The difference $\ell\Omega$ between the frequencies of the diffracted and undiffracted light represents a Doppler shift due to the motion of the acoustic wavefronts (Appendix B). The direction of propagation of the ℓth-order beam, $\vec{k}_\ell/k_\ell$, makes an angle θ_ℓ with the z axis, where for small angles

$$\theta_\ell \simeq \tan\theta_\ell = \frac{\ell K}{k} = \frac{\ell\lambda}{\Lambda}. \tag{12}$$

Its amplitude U_ℓ and intensity I_ℓ are given by

$$U_\ell = J_\ell(\Delta\phi) \tag{13}$$

and

$$I_\ell = J_\ell^2(\Delta\phi), \tag{14}$$

respectively. These are shown for the first few orders as a function of $\Delta\phi$ in Fig. 3. Note that the zero order disappears when $\Delta\phi \simeq 2.4$ radians. It is evident that if we reverse the direction of sound propagation, each positive order is replaced by the corresponding negative order and vice versa.

We assumed above that the sound column acts like a sinusoidal phase grating moving with a velocity V. This assumption holds if the interaction length L is sufficiently small. For large L the diffraction spectrum disappears due to destructive interference.[6,8] To see why, let us divide the interaction length L into a number of small elements dL as indicated in Fig. 4. The phase modulation $d\phi$ introduced by the first element (left edge of the sound column) is so small that only the $\ell = \pm 1$ orders are generated, i.e., plane waves at angles $\pm\lambda/\Lambda$ to the incident wave. The three waves now enter the next element where they again split generating new orders and modifying the amplitude and phase of the $\ell = 0, \pm 1$ orders. This procedure is carried on to the last element (right edge). The various orders so produced appear with the same frequencies and at the same angles as given by Eq. (8). Their amplitudes, however, do not continue to grow but return to zero as L increases.

Consider the positive first order; its wavefront is tilted at an angle λ/Λ with respect to that of the incident light. Contributions to this order generated at different points along an acoustic wavefront do not add in phase. When $L \simeq 2\Lambda^2/\lambda$, these contributions cancel each other completely since the path difference (OA-OB) between contributions from the extreme ends of the acoustic wavefront is λ. Note that we have neglected the difference in wavelengths between the zero and first order waves since $\Omega \ll \omega$. The higher orders, owing to their larger angles of tilt, disappear for even smaller values of L. For example, the second order disappears when $L = \Lambda^2/2\lambda$. Thus, when $L = 2\Lambda^2/\lambda$, or larger, all the light goes into the zero order. This value of L may be called the transition length:

$$L_t = 2\Lambda^2/\lambda. \qquad \textit{Transition Length} \tag{15}$$

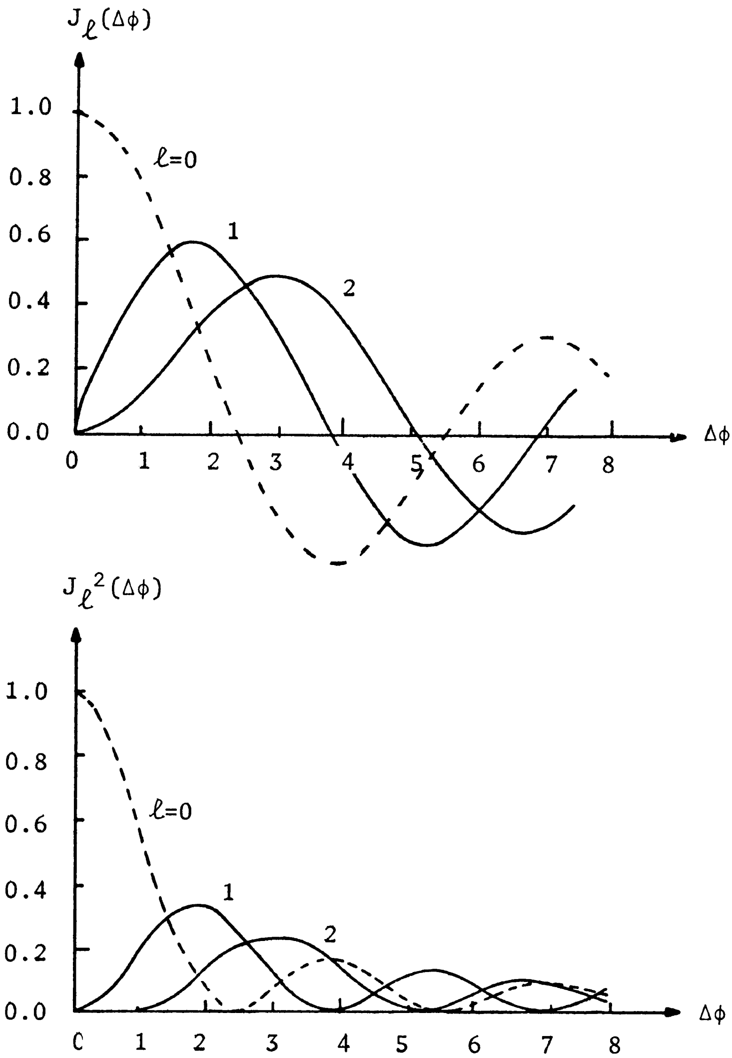


Fig. 3. Raman-Nath diffraction for normal incidence. J_ℓ and J_ℓ^2 are the amplitude and intensity of the ℓth order of diffraction. Since $J_\ell^2 = J_{-\ell}^2$, the intensities of the corresponding positive and negative orders are equal. $\Delta\phi$ is the magnitude of phase modulation produced by the sound beam.

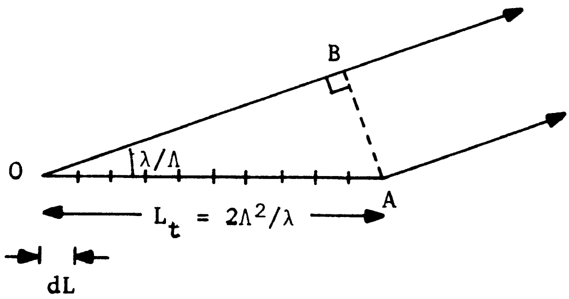
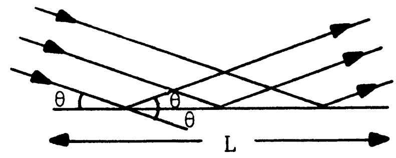


Fig. 4. The transition length L_t. *Destructive interference of contributions to the positive first order* ($\ell = +1$) *generated at different points along an acoustic wavefront* OA. *The path difference* OA - OB = λ.

It is clear that for light to diffract, contributions along an acoustic wavefront must interfere constructively. This can happen for the positive first order if the light is incident at an angle $\lambda/2\Lambda$ as shown in Fig. 5a. The positive first-order waves, still going off at an angle λ/Λ with respect to the incident light, now appear to be reflected from the acoustic wavefronts as if they were (partially reflecting) mirrors. No matter how long we make L, all points on an acoustic wavefront contribute to the positive first order in phase. This cumulative interaction occurs only for the positive first order; the negative first and all higher orders are still subject to destructive interference.

Thus, depending on the geometry of the experiment, one may observe several positive and negative diffraction orders or only one of the two first orders (besides the zero order). When the interaction length L is small and the acoustic frequency Ω is low so that several orders are observed, the diffraction process is called the Raman-Nath diffraction, or is said to be in the Raman-Nath region (regime), in honor of the first authors to successfully explain a variety of phenomena that occur in this region. However, when L is large and Ω is high, the diffraction process is called the Bragg diffraction or is said to be in the Bragg region (regime), in analogy to the selective diffraction of x rays by lattice planes of crystals first observed by Bragg. The angle $\theta \simeq \lambda/2\Lambda$, or more precisely, $\sin\theta = \lambda/2\Lambda$, is called the Bragg angle. From Fig. 5b we see that when light is incident at the Bragg angle, rays reflected from two acoustic wavefronts remain in phase since their path difference is equal to λ:

$$2\Lambda \sin\theta = \lambda. \qquad \textit{Bragg Equation} \qquad (16)$$



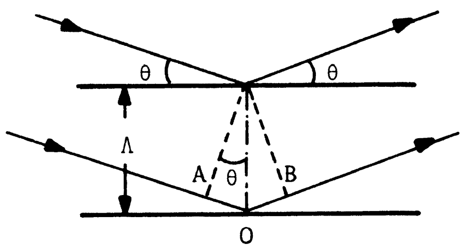


Fig. 5. Bragg diffraction. (a) When light is incident at an angle $\theta \simeq \lambda/2\Lambda$*, the positive first order light appears to be reflected from an acoustic wavefront and the contributions generated at different points are in phase. (b) Rays reflected from two acoustic wavefronts remain in phase. The path difference* AO + OB = $2\Lambda \sin\theta = \lambda$. *Angle* θ *is called the Bragg angle, later denoted by* Θ.

Equation (16) is called the Bragg equation. The angle $\theta = \sin^{-1} (\lambda/2\Lambda)$ is called the Bragg angle, and we shall denote it by Θ.

To determine the intensity of the Bragg-diffracted light, let the amplitude of phase modulation produced by an element of length dL be

$$d\phi = k \frac{\Delta n}{n} \frac{dL}{\cos\Theta} , \tag{17}$$

which varies in the x direction as $\sin(\Omega t - Kx)$. The complex amplitude

$$U_0 \; e^{i(\omega t - \vec{k}\cdot\vec{r})}$$

of the incident light (zero order) is modified after passing through this element to

$$U_0\, e^{i(\omega t-\vec{k}\cdot\vec{r})}\, e^{id\phi\,\sin(\Omega t-Kx)} \simeq U_0\, e^{i(\omega t-\vec{k}\cdot\vec{r})}[1 + id\phi\,\sin(\Omega t-Kx)]$$

$$= U_0\, e^{i(\omega t-\vec{k}\cdot\vec{r})} + \tfrac{1}{2}U_0 d\phi\, e^{i[(\omega+\Omega)t-\vec{k}\cdot\vec{r}-Kx]}$$

$$- \tfrac{1}{2}U_0 d\phi\, e^{i[(\omega-\Omega)t-\vec{k}\cdot\vec{r}+Kx]}. \qquad (18)$$

The last two terms on the right-hand side of this equation are the positive and negative first orders that diverge from the zero order due to their spatial frequency components $K/2\pi$ in the x direction. As explained above, only the positive first order grows; the negative first order is eliminated by destructive interference. The amplitude of the positive first order is given by

$$dU_{+1} = \tfrac{1}{2}U_0 d\phi, \qquad (19)$$

and its rate of growth with distance L is simply

$$\frac{dU_{+1}}{dL} = \tfrac{1}{2}U_0 \frac{d\phi}{dL}. \qquad (20)$$

The conservation of energy requires that the intensity of the incident light be equal to the sum of the intensities of the zero and positive first orders, i.e.,

$$1 = U_0^2 + U_{+1}^2. \qquad (21)$$

Differentiation of this equation gives

$$\frac{dU_0}{dL} = -\frac{U_{+1}}{U_0}\frac{dU_{+1}}{dL} = -\tfrac{1}{2}U_{+1}\frac{d\phi}{dL}. \qquad (22)$$

The last step is obtained by substituting Eq. (20). Solving Eqs. (20) and (22) subject to the boundary conditions

$$U_0(0) = 1$$

$$U_{+1}(0) = 0, \qquad (23)$$

we obtain

$$U_{+1} = \sin\frac{\Delta\phi}{2} \qquad (24a)$$

$$U_0 = \cos\frac{\Delta\phi}{2} \qquad (24b)$$

where

$$\Delta\phi = \frac{d\phi}{dL} L$$

$$= k \frac{\Delta n}{n} \frac{L}{\cos\Theta} \qquad (25)$$

is the magnitude of phase modulation produced by the entire sound column of width L. When $\Delta\phi = (2m+1)\pi$, where m is a positive integer, all the incident light is diffracted, as shown in Fig. 6.

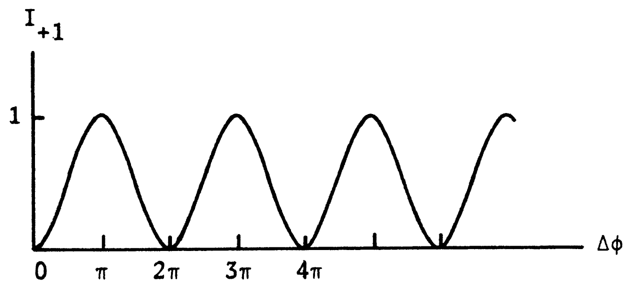


Fig. 6. Intensity of Bragg-diffracted light $I_{+1} = \sin^2(\Delta\phi/2)$ *where the phase modulation* $\Delta\phi = k(\Delta n/n)(L/\cos\Theta)$. *Light of unit intensity is assumed to be incident at the Bragg angle* Θ. *Intensity of undiffracted (zero order) light is* $I_0 = \cos^2(\Delta\phi/2)$.

For large L and high Ω we have shown that unless the light is incident at the Bragg angle, it will not be diffracted. This would be strictly true if there were only one sound wave present in the diffracting medium. However, any wave limited in extent can be represented as a sum of plane waves of varying amplitude traveling in different directions. As a consequence, the Bragg condition is relaxed, i.e., even when the light is incident at an angle slightly off the Bragg angle, part of it will be diffracted. For a uniform sinusoidal sound beam of width L, the relative amplitude of the plane wave component traveling at an angle $\Delta\theta$ with respect to the x axis is given by

$$\text{sinc}\left(\frac{L \sin\Delta\theta}{\Lambda}\right) \simeq \text{sinc}\left(\frac{L\Delta\theta}{\Lambda}\right) , \qquad (26)$$

where $\text{sinc } x = \sin(\pi x)/\pi x$. Hence, when the light is incident at an angle different from the Bragg angle by $\Delta\theta$, a fraction

$$U_{+1} = \sin\left[\frac{\Delta\phi}{2} \text{sinc}\left(\frac{L\Delta\theta}{\Lambda}\right)\right] \qquad (27)$$

will be diffracted. The angle between the diffracted and the zero order beams, i.e., the angle of deflection, is always equal to twice the Bragg angle. Thus when the angle of incidence is increased by $\Delta\theta$, the angle of diffraction is decreased by the same amount. The amount of diffracted light is reduced by a factor of $\mathrm{sinc}^2(L\Delta\theta/\Lambda)$, assuming $\sin(\Delta\phi/2) \approx (\Delta\phi/2)$ for small $\Delta\phi$.

From the foregoing, it should be clear that by measuring the amount of diffracted light as a function of the angle of incidence, one can determine the angular distribution of the acoustic energy.[9] In practice, this may be done in four different, but equivalent, ways. In the following, $\Delta\phi$ is assumed to be small.

1. Change the angle of incidence by tilting the incident beam as shown in Fig. 7. If the angle of incidence is $\theta = \Theta + \Delta\theta$, the angle of diffraction (observation) is given by

$$\theta_{+1} = 2\Theta - \theta = \Theta - \Delta\theta. \tag{28}$$

For a uniform sound beam, the diffracted intensity is given by

$$\frac{I(\Delta\theta)}{I(0)} = \mathrm{sinc}^2\left(\frac{L\Delta\theta}{\Lambda}\right) \tag{29}$$

and is shown as a function of $\Delta\theta$ in Fig. 8.

2. Rotate the sound column (about y axis) by an angle $\Delta\theta$ keeping the source and detector fixed as indicated in Fig. 9. The diffracted intensity observed for a uniform sound beam in this case is also given by Eq. (29). The sound column can be rotated by rotating the diffraction system. Mechanical rotation can, however, be avoided if the transducer is designed to steer the acoustic beam. A brief discussion of acoustic beam steering is given in Appendix C.

3. Keep the source and the sound column fixed but change the acoustic frequency from Ω to $\Omega + \Delta\Omega$. This implies a change of

$$\Delta K = \frac{\Delta\Omega}{V} \tag{30}$$

in the acoustic wavenumber and a change of

$$\Delta\Theta = \frac{\Delta K}{2k}\,\frac{1}{\cos\Theta} = \frac{\Delta K}{K}\,\tan\Theta \tag{31}$$

in the Bragg angle. The intensity of the diffracted beam is again given by Eq. (29) with $\Delta\Theta$ substituted for $\Delta\theta$, i.e.,

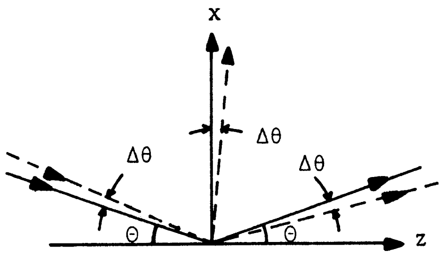


Fig. 7. When light is incident at an angle Θ + Δθ, *Bragg diffraction is caused by the sound wave component traveling in a direction making an angle* Δθ *with the x axis. For a uniform sound beam of small amplitude, the intensity of diffracted light is reduced by a factor of* $\text{sinc}^2(L\Delta\theta/\Lambda)$. *The angle between the undiffracted and diffracted light is always* 2Θ.

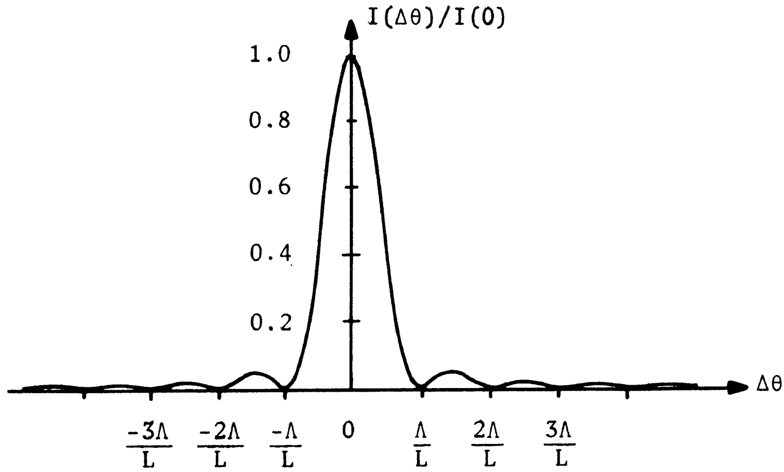


Fig. 8. Bragg intensity as a function of the relative angle of incidence Δθ = θ - Θ.

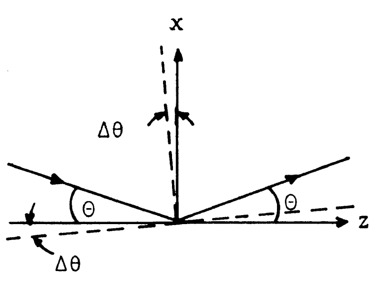


Fig. 9. When sound column is rotated by Δθ, *the central sound component is also rotated by* Δθ. *Diffraction is caused by the component inclined to the central at an angle* Δθ, *i.e., the one traveling parallel to the* x *axis. The diffracted beam is observed in the same direction as when* Δθ = 0, *but its intensity, for a uniform sound beam of small amplitude, is reduced by a factor of* $\text{sinc}^2(L\Delta\theta/\Lambda)$.

$$\frac{I(\Delta K)}{I(0)} = \mathrm{sinc}^2\left(\frac{\Delta K}{2\pi} L \tan\Theta\right). \qquad (32)$$

Its direction, however, makes an angle of $2(\Theta + \Delta\Theta)$ with the zero-order beam as shown in Fig. 10. Thus by changing the acoustic frequency the light can be diffracted in any direction; the larger the change in frequency, the larger the change in angle of deflection but the lower the diffracted intensity. The intensity of the diffracted beam can be increased to its maximum value if the sound column is rotated by $\Delta\Theta$ so that the Bragg condition is satisfied for the new acoustic frequency.

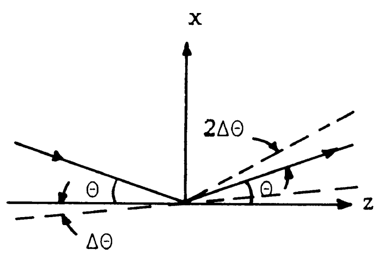


Fig. 10. When the acoustic frequency is changed by $\Delta\Omega$, the Bragg angle changes by $\Delta\Theta = (\Delta\Omega/\Omega)\tan\Theta$. Diffraction is caused by the sound wave component whose wavefronts make an angle of $\Theta + \Delta\Theta$ with the incident light. Light is diffracted in a direction making an angle of $2(\Theta + \Delta\Theta)$ with the zero order light, and its intensity for a uniform sound beam of small amplitude is reduced by a factor of $\mathrm{sinc}^2(L\Delta\Theta/\Lambda)$ *compared to when $\Delta\Omega = 0$. Note that the direction of incident light is held fixed.*

4. Similarly, if the angle of incidence changes from Θ to $\Theta + \Delta\theta$, the direction of the diffracted beam can be held fixed by changing the acoustic frequency from Ω to $\Omega + \Delta\Omega$ where $\Delta\Omega = kV\Delta\theta$ and its intensity can attain its maximum value if the sound column is also rotated by an angle $\Delta\Theta = \Delta\theta/2$. This can be seen from Fig. 11. Moreover, if the phase of the sound wave is adjusted by Φ, a phase of this amount is also introduced in the diffracted wave (see Eqs. (1), (6), and (8), or equivalently, Eq. (18)). In other words, no matter what the direction or phase of the incident light, it can be diffracted in any direction and with any phase by changing the frequency and phase of the injected sound wave, respectively.

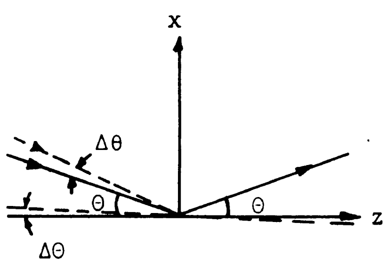


Fig. 11. When the angle of incidence changes by $\Delta\theta$, the direction of the diffracted beam is held fixed by changing the Bragg angle by $\Delta\Theta = \Delta\theta/2$ through a change in the acoustic frequency by $\Delta\Omega = (\Omega\Delta\Theta)(1/\tan\Theta)$. Diffraction is caused by the sound wave component whose wavefronts make an angle of $\Theta + \Delta\Theta$ with the incident light. Intensity of the diffracted beam is reduced by a factor of $\mathrm{sinc}^2(L\Delta\theta/2\Lambda)$ *compared to when $\Delta\theta = 0$.*

It may be added that if the direction of the sound wave is reversed, the direction of the diffracted wave remains unchanged. Its frequency, however, changes from $\omega + \Omega$ to $\omega - \Omega$.

All the angles, the wavenumbers (or wavelengths) referred to above are measured within the diffracting medium. Their counterparts outside the medium, i.e., in air (vacuum), can be determined by using Snell's law. Thus if the sides of the sample carrying the acoustic wave are parallel to the direction of sound propagation, Snell's law gives

$$\sin\theta_a = n \sin\theta \tag{33}$$

where the subscript a stands for air (vacuum).

3. PARTICLE PICTURE OF RAMAN-NATH AND BRAGG DIFFRACTION

In principle, one may conceive of the diffraction of light by sound as a three-particle scattering that may be repeated (multiple scattering) depending on the parameters of an experiment.[10] In the particle picture, the light beam with propagation vector $\vec{k}$ and frequency ω is considered to consist of a stream of particles (photons) with momentum $\hbar\vec{k}$ and energy $\hbar\omega$. The sound beam, likewise, can be thought of as made up of particles (phonons) with momentum $\hbar\vec{K}$ and energy $\hbar\Omega$.

Three-particle scattering in which a phonon is absorbed is illustrated in Fig. 12. The direction of $\vec{K}$ is reversed when a phonon is generated. To distinguish between incident and scattered photons, the parameters of the latter are denoted with a prime. As in the previous section, the vectors $\vec{k}$, $\vec{K}$, and $\vec{k}'$ are assumed to lie in the zx plane. The absorption or emission of a phonon is described by the conservation of energy and momentum:

$$\left.\begin{aligned} \omega + \Omega &= \omega' \\ \vec{k} + \vec{K} &= \vec{k}' \end{aligned}\right\} \qquad \textit{Phonon Absorption} \tag{34}$$

$$\left.\begin{aligned} \omega &= \Omega + \omega' \\ \vec{k} &= \vec{K} + \vec{k}' \end{aligned}\right\}. \qquad \textit{Phonon Emission} \tag{35}$$

Since $\Omega << \omega$, we may assume $\omega \simeq \omega'$, and therefore $k \simeq k'$ for isotropic materials. Regardless of whether a phonon is absorbed or emitted, the conservation of momentum can be written in terms of its components as

$$k \cos\theta = k' \cos\theta'$$

and

$$-k \sin\theta + K = k' \sin\theta'. \tag{36}$$

Since $k = k'$, the first of these equations yields $\theta = \theta'$. The second equation, therefore, gives

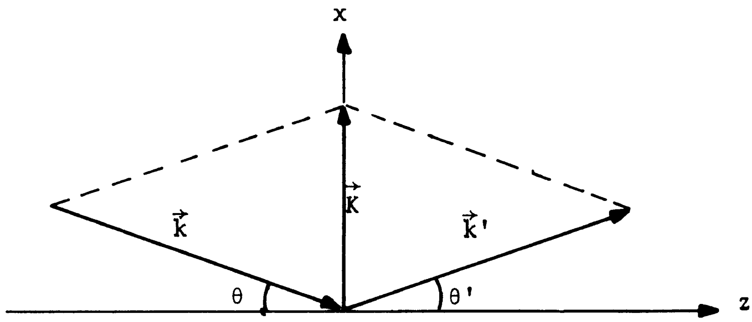


Fig. 12. Momentum conservation for photon-phonon interaction in which a phonon is annihilated. Direction of $\vec{K}$ *is reversed when a phonon is created.*

$$\sin\theta = \frac{K}{2k} = \frac{\lambda}{2\Lambda} . \tag{37}$$

This is the Bragg equation (16) obtained earlier. Note that it limits the sound wavelength, which may be scattered to a value greater than $\Lambda = \lambda/2$. The case $\Lambda = \lambda/2$ corresponds to back scattering, i.e., $\vec{k}' = -\vec{k}$.

So far, we have treated the diffraction of light by sound as a sum of separate three-particle interactions. Let us now consider the effects of finite widths of light and sound beams. It was pointed out earlier that a wave restricted in size spreads due to diffraction. The angular spread for a uniformly illuminated rectangular aperture can be characterized by the angle from the normal at which the first zero of the far-field diffraction pattern occurs. Thus we can write the angular width of light and sound beams of width W and L, respectively, as

$$\delta = 2\,\frac{\lambda}{W} \qquad \textit{Angular Width of k-Vector} \tag{38}$$

and

$$\Delta_s = 2\,\frac{\Lambda}{L} \cdot \qquad \textit{Angular Width of K-Vector} \tag{39}$$

The subscript s denotes sound and is used to avoid confusion with the variation Δ in n, θ, Ω, etc. Because of the spread in vectors $\vec{k}$ and $\vec{K}$, the three-particle Bragg condition is relaxed and becomes an angular distribution of three-particle interactions, thereby permitting scattering to occur at various angles.

The angular spread in K-vector (if sufficient) can also cause multiple (i.e., successive) interactions. The conservation of momentum for a two-stage interaction is shown in Fig. 13. The energy conservation enters into this figure by the requirement that

$$k = k' = k''$$

and

$$\theta = \theta' = \theta'' = \theta''' = \Theta. \qquad (40)$$

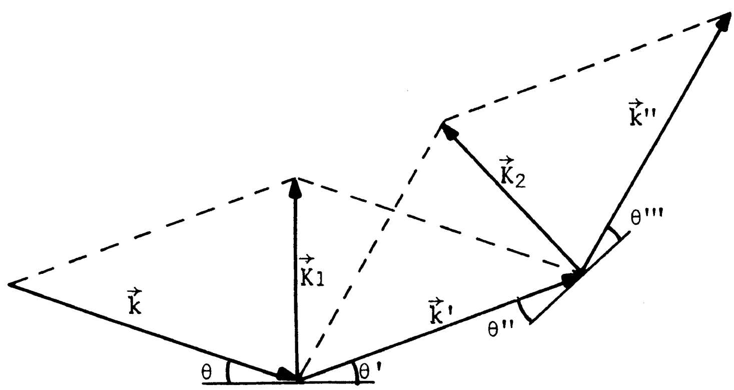


Fig. 13. Momentum conservation for successive scattering of a photon $\vec{k}$ by two phonons $\vec{K}_1$ and $\vec{K}_2$.

We note from the figure that the conservation of energy and momentum can occur in a two-step process only if there are two phonons with their K-vectors properly aligned. Moreover, the smaller the Bragg angle, the smaller the spread in K-vectors required for multiple scattering. In other words, multiple scattering is possible at low acoustic frequencies (since Θ is then small), and energy and momentum conservation cannot both occur if multiple scattering takes place at sufficiently high acoustic frequencies. In this sense multiple scattering becomes a forbidden process at high acoustic frequencies.

Consequently, we can distinguish the region for which multiple scattering is highly probable (Raman-Nath region) from the one for which it is relatively improbable (Bragg region). Multiple scattering will readily occur if the angular half width of K-vectors is large compared to the Bragg angle and will not readily occur when the reverse is true. The limit between the two regions can be expressed by the equation

$$\tfrac{1}{2}\Delta_s = \Theta. \qquad (41)$$

Or, using $\Theta \simeq \sin\Theta = \lambda/2\Lambda$ and $\Delta_s = 2\Lambda/L$, we get

$$L = 2\Lambda^2/\lambda. \tag{42}$$

This is the transition length L_t defined in Eq. (15).

In the remaining part of this section, we shall assume high acoustic frequencies so that the scattering is always in the Bragg region. For $\Omega << \omega$, we assume $\omega = \omega'$ and therefore $k = k'$ and make the elementary but useful construction shown in Fig. 14, which is drawn in momentum space.[11] The locus of the scattering interaction in this space is a circle of radius k. The scattering (or deflection) angle is 2Θ where Θ is the Bragg angle given by the Bragg equation

$$\sin\Theta = \frac{K}{2k}. \tag{43}$$

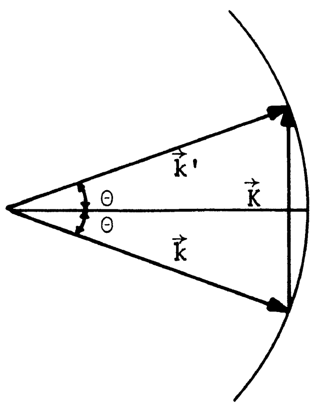


Fig. 14. Vector diagram for Bragg interaction of plane monochromatic optical and acoustic waves. If the direction of $\vec{k}$ or $\vec{K}$ is changed, thereby changing the angle of incidence Θ, the vector sum $\vec{k} + \vec{K}$ no longer falls on the circle, and the intensity of the scattered wave becomes zero.

If we assume that neither the optical nor the acoustic beam has any angular width (i.e., if they are both plane waves), only a phonon of precisely the correct $\vec{K}$ will scatter $\vec{k}$. If the direction of $\vec{k}$ or $\vec{K}$ is changed, thereby changing the angle of incidence Θ, the vector sum $\vec{k} + \vec{K}$ no longer falls on the circle and, therefore, the intensity of the scattered (diffracted) light goes to zero.

To appreciate the effect of nonzero diffraction spread, first consider the case in which the angular width of the acoustic beam is larger than that of the optical beam, i.e., $\Delta_s > \delta$. The sound wavevector $\vec{K}$ has a well defined magnitude but an angular width of Δ_s. If δ is zero, only one K-vector (which is properly aligned with respect to the incident k-vector) can contribute to the scattered light. For a finite δ, all the K-vectors in the angular range δ about the vertical scatter light. Consequently, the angular width of the scattered light is also δ as illustrated in Fig. 15. Since its wavelength is also the same as that of the incident light ($k' = k$), we see that the coherence of the light beam, both spatial and temporal, is preserved. Note that only a portion of the acoustic energy is useful in scattering.

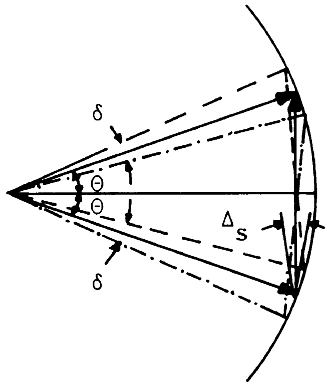


Fig. 15. Vector diagram for Bragg diffraction of an optical beam of angular width δ by an acoustic beam of angular width $\Delta_S > \delta$. All the k-vectors in the angular range δ are diffracted by the corresponding K-vectors in the range δ. The remaining K-vectors do not take part in the acousto-optic interaction, and therefore only a portion of the acoustic energy is useful in scattering. Note that the angular width of the diffracted beam is the same as that of the incident beam.

We now consider the other case, i.e., the one in which the diffraction spread of the optical beam in larger than that of the acoustic beam ($\delta > \Delta_S$). Figure 16 illustrates that in this case only a portion of the incident light is scattered, and the angular width of the scattered light is equal to Δ_S, i.e., the scattered beam is wider than the incident beam. Thus the angular width of the scattered light always corresponds to the smaller of the two angles δ and Δ_S.

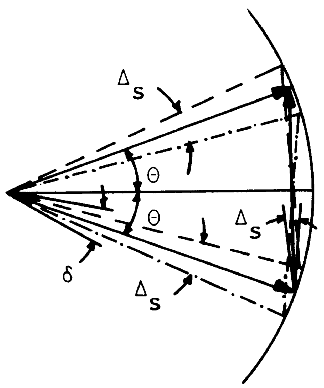


Fig. 16. Vector diagram for Bragg diffraction when $\delta > \Delta_S$. Only the k-vectors in the angular range Δ_S are diffracted, and therefore a portion of the incident light takes part in scattering. The angular width of the scattered light is equal to Δ_S and, therefore, smaller than that of the incident light.

Finally, we consider the effect of variation in acoustic frequency. In what follows we shall neglect the diffraction spread δ. If the direction of the incident light changes, that of the scattered light can be held fixed by an appropriate change in the acoustic frequency as shown in Fig. 17. When the angle of incidence increases from Θ to Θ + Δθ, the direction of the scattered light remains unchanged if the acoustic frequency is increased by

$$\Delta\Omega = V\Delta K = V \frac{K}{2 \tan\Theta} \Delta\theta . \quad (44)$$

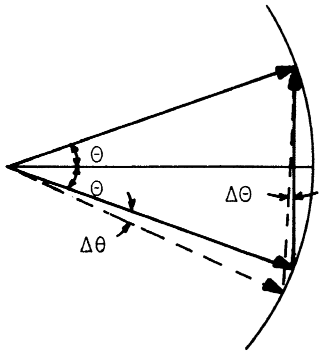


Fig. 17. Vector diagram for Bragg diffraction. When the angle of incidence changes by $\Delta\theta$*, the diffracted beam is held fixed by changing the Bragg angle by* $\Delta\Theta = \Delta\theta/2$ *through a change in the acoustic frequency by* $\Delta\Omega = \Omega\Delta\Theta/\tan\Theta$*. Intensity of the diffracted beam remains unchanged if the sound column is also rotated by* $\Delta\Theta$*. Diffraction spread* δ *of the optical beam is assumed zero.*

An increase in the frequency by $\Delta\Omega$ changes the Bragg angle from Θ to $\Theta + \Delta\Theta$ where $\Delta\Theta = \Delta\theta/2$. The intensity of the scattered light for a uniform sound beam of width L will be less compared to when $\Delta\theta = 0$ by a factor of $\text{sinc}^2(L\Delta\Theta/\Lambda)$ for small $\Delta\phi$. It can be increased to its maximum value if the sound column is rotated by an angle $\Delta\Theta$, thus satisfying the Bragg condition at the new frequency.

If, however, the direction of the incident light is fixed, a change in the acoustic frequency can be detected by the corresponding change in the direction (and intensity) of the scattered light. We see from Fig. 18 that as the frequency changes from Ω to $\Omega + \Delta\Omega$, the direction of the scattered light changes by $2\Delta\Theta$ (where $\Delta\Theta$ is the change in the Bragg angle corresponding to a change $\Delta\Omega$ in the frequency), i.e., the scattering angle changes from 2Θ to $2(\Theta + \Delta\Theta)$.

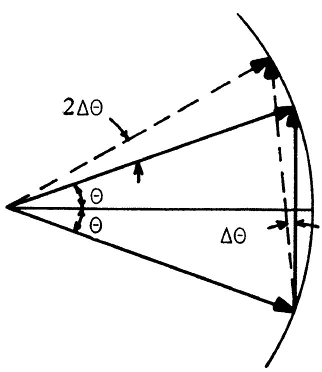


Fig. 18. Vector diagram for Bragg diffraction. A change of $\Delta\Omega$ *in the acoustic frequency can be detected by measuring the change* $2\Delta\Theta$ *in the direction of the diffracted beam where* $\Delta\Theta = (\Delta\Omega/\Omega)\tan\Theta$ *is the change in Bragg angle corresponding to a change* $\Delta\Omega$ *in the frequency. Note that the direction of incident light is held fixed. Diffraction spread* δ *of the optical beam is assumed zero.*

4. PARTIAL WAVE ANALYSIS OF DIFFRACTION OF LIGHT BY SOUND

Consider a plane wave of light of frequency ω incident at an angle θ from the z axis on a medium in which a sound wave of frequency Ω is traveling in the positive x direction with a velocity V. The time and spatial dependence of the refractive index of the sound column can be written as

$$\begin{aligned} n(x,t) &= n + \Delta n(x,t) \\ &= n + \Delta n \sin(\Omega t - Kx + \Phi), \end{aligned} \qquad (45)$$

where n is the refractive index of the medium in the absence of the sound wave, and $\Delta n(x,t)$ represents the index wave of amplitude Δn, phase constant Φ, and wavenumber $K = \Omega/V$.

From Maxwell's equations, the optical wave equation describing the propagation of the electric field $\vec{E}$ in a nonmagnetic nonconducting medium can be written in the form

$$\nabla^2\vec{E} - \nabla(\nabla \cdot \vec{E}) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\vec{E}\left[1 + 2\,\frac{\Delta n(x,t)}{n}\right]$$

Optical Wave Equation (46)

where $v = c/n$ is the speed of light in the medium. Substituting for $\Delta n(x,t)$ from Eq. (45) we can write it as

$$\nabla^2\vec{E} - \nabla(\nabla \cdot \vec{E}) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\vec{E}\left\{1 + \frac{\Delta n}{in}\left[e^{i(\Omega t - Kx + \Phi)} - e^{-i(\Omega t - Kx + \Phi)}\right]\right\} \qquad (47)$$

In the partial wave analysis, one assumes that the diffracted light is given by

$$\vec{E}(\vec{r}) = \tfrac{1}{2} \sum_{\ell=-\infty}^{\infty} \vec{U}_\ell(z)\, e^{i(\omega_\ell t - \vec{k}_\ell \cdot \vec{r})} + \text{c.c.}, \tag{48}$$

where

$$\omega_\ell = \omega + \ell\Omega, \tag{49}$$

and

$$\vec{k}_\ell = \vec{k} + \ell\vec{K}$$

$$= (-k \sin\theta + \ell K,\ 0,\ k \cos\theta) \tag{50}$$

and c.c. denotes the complex conjugate. Subject to the initial conditions

$$U_\ell(z=0) = \delta_{\ell 0} \tag{51}$$

where $\delta_{\ell 0}$ is the Kronecker δ, Eq. (48) correctly represents the unit amplitude incident field. The sum in this equation represents a series of diffracted plane waves of frequencies ω_ℓ, wavevectors $\vec{k}_\ell$, and amplitudes $\vec{U}_\ell(z)$ that vary within the sound column along the z axis. The direction of propagation of the ℓth partial wave, as indicated in Fig. 19, is given by

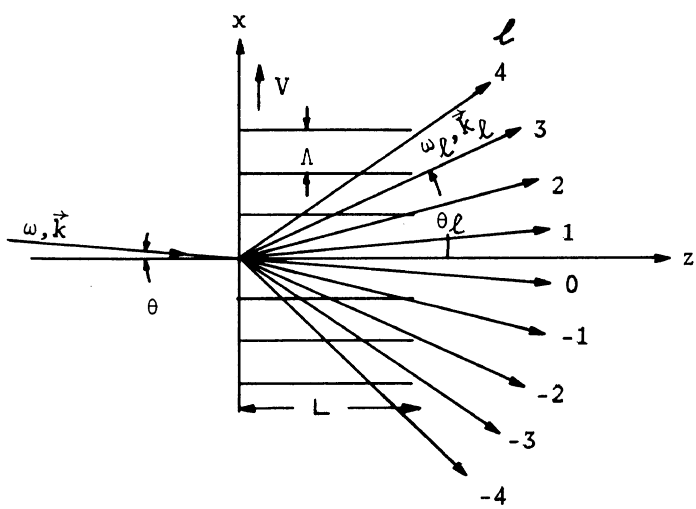


Fig. 19. Diffraction of light by sound. Light incident at an angle θ *from the z axis interacts with a sound wave traveling in the positive x direction. The frequency and wavevector of the ℓth diffracted wave are given by* $\omega_\ell = \omega + \ell\Omega$ *and* $\vec{k}_\ell = \vec{k} + \ell\vec{K}$, *respectively, where* ω *and* $\vec{k}$ *are the frequency and wavevector of the incident light. The angle* θ_ℓ *is given by Eq. (52),* $\tan\theta_\ell = -\tan\theta + (\ell K/k \cos\theta)$.

$$\tan\theta_\ell = \frac{k_{\ell x}}{k_{\ell z}} = \frac{-k\sin\theta + \ell K}{k\cos\theta} = -\tan\theta + \frac{\ell K}{k\cos\theta}\,. \tag{52}$$

Here θ_ℓ is the angle between $\vec{k}_\ell$ and the z axis. For small angles, the angle between two successive diffraction orders is given, approximately, by λ/Λ. In the partial wave expansion assumed above, each plane wave except $\vec{U}_0$ originates from the absorption or emission of one or more phonons by the incident light in the region of interaction. The quantum nature of the interaction allows only a discrete number of possible directions and frequencies for the diffracted wave components. The partial wave expansion in Eq. (48) clearly takes this fact into account.

Substituting for $\vec{E}$ from Eq. (48) in Eq. (47) and assuming that $\vec{k}_\ell \cdot \vec{U}_\ell = 0$, i.e., $\vec{E}_\ell$ is transverse, we get

$$\tfrac{1}{2}\sum_{\ell=-\infty}^{\infty}\left[\frac{d^2U_{\ell x}}{dz^2}\hat{x} + \frac{d^2U_{\ell y}}{dz^2}\hat{y} - k_\ell^2\vec{U}_\ell - i\left(\vec{k}_\ell\frac{dU_{\ell z}}{dz} - 2k_{\ell z}\frac{d\vec{U}_\ell}{dz}\right)\right]$$

$$\cdot\, e^{i(\omega_\ell t - \vec{k}_\ell\cdot\vec{r})} + \text{c.c.}$$

$$= \frac{1}{v^2}\,\frac{1}{2}\sum_{\ell=-\infty}^{\infty}(i\omega_\ell)^2\left[\vec{U}_\ell + \frac{\Delta n}{in}\left(\vec{U}_{\ell-1}e^{i\Phi} - \vec{U}_{\ell+1}\,e^{-i\Phi}\right)\right]$$

$$\cdot\, e^{i(\omega_\ell t - \vec{k}_\ell\cdot\vec{r})} + \text{c.c.} \tag{53}$$

If the amplitude of each diffracted plane wave increases slowly with distance z (so that the amplitude change is small in one optical wavelength) the terms containing $d^2U_{\ell x}/dz^2$ and $d^2U_{\ell y}/dz^2$ may be neglected. Comparing coefficients of

$$e^{i(\omega_\ell t - \vec{k}_\ell\cdot\vec{r})}$$

and rearranging terms, we get from Eq. (53)

$$\vec{k}_\ell\frac{dU_{\ell z}}{dz} - 2k_{\ell z}\frac{d\vec{U}_\ell}{dz} + \frac{\Delta n}{n}\frac{\omega_\ell^{\,2}}{v^2}\left(\vec{U}_{\ell-1}e^{i\Phi} - \vec{U}_{\ell+1}e^{-i\Phi}\right)$$

$$= -i\left(\frac{\omega_\ell^{\,2}}{v^2} - k_\ell^{\,2}\right)\vec{U}_\ell. \tag{54}$$

It may be mentioned here that it is not necessary to assume Eqs. (49) and (50) in advance. When we substitute Eq. (48) in Eq. (47) and compare the coefficients of each exponential, Eqs. (49) and (50) follow naturally.

If we resolve the incident E-vector into two components that are parallel and perpendicular to the scattering plane zx, we may assume that, for each of the diffracted waves, the polarization of these components does not change as they traverse the medium. Calculating the dot product of Eq. (54) with $\hat{U}_\ell$, where $\hat{U}_\ell$ is a unit vector in the direction of $\vec{U}_\ell$, we obtain

$$\frac{dU_\ell}{dz} + \frac{\Delta n}{2nk\cos\theta}\,\frac{\omega_\ell^2}{v^2}\left(\hat{U}_\ell \cdot \hat{U}_{\ell-1}U_{\ell-1}\,e^{i\Phi} - \hat{U}_\ell \cdot \hat{U}_{\ell+1}U_{\ell+1}\,e^{-i\Phi}\right)$$

$$= -\frac{i}{2k\cos\theta}\left(\frac{\omega_\ell^2}{v^2} - k_\ell^2\right)U_\ell. \qquad (55)$$

In Eq. (55) we have substituted $k_{\ell z} = k\cos\theta$ from Eq. (50). If we had started with the scalar wave equation for the optical field, the amplitudes $\vec{U}_\ell$ would have been scalar quantities, and therefore the factor containing the dot products in Eq. (55) would be absent. If the incident field is polarized perpendicular to the scattering plane, then the field in any diffraction order is also polarized in this direction and, therefore,

$$\hat{U}_\ell \cdot \hat{U}_{\ell-1} = \hat{U}_\ell \cdot \hat{U}_{\ell+1} = 1. \qquad \textit{Perpendicular Polarization} \qquad (56)$$

If it is polarized in the scattering plane, then the angle between the fields in two diffraction orders is equal to the angle between their corresponding propagation vectors. Since the angle between two successive orders is approximately constant and equal to λ/Λ

$$\hat{U}_\ell \cdot \hat{U}_{\ell-1} \simeq \hat{U}_\ell \cdot \hat{U}_{\ell+1} \simeq \cos(\lambda/\Lambda). \qquad \textit{Parallel Polarization} \qquad (57)$$

At this point it should be mentioned that a sound wave propagating in a liquid produces an index wave whose amplitude is independent of the optical polarization. However, this is not true in the case of solids (even if they are isotropic). Moreover, whereas only longitudinal waves can propagate in a liquid, both longitudinal and transverse waves can propagate in a solid (simultaneously or separately depending on the method of excitation and the boundary conditions). In solids, therefore, the index variations produced by a sound wave depend on optical and acoustic polarizations. For simplicity we shall assume that the index variation is independent of optical polarization. Accordingly the following discussion is strictly valid only for liquid diffracting media. The corresponding results for solid media can be obtained by considering the appropriate value of the index variation.

For arbitrary polarization, we can find the diffracted fields corresponding to the perpendicular and parallel components of the incident field and add the two to get the resultant field in each order. Thus In Eq. (55) we can assume that $\hat{U}_\ell \cdot \hat{U}_{\ell-1} = \hat{U}_\ell \cdot \hat{U}_{\ell+1}$ and write the equation in the form

$$\frac{dU_\ell}{dz} + \frac{\Delta n}{2nk\cos\theta}\,\frac{\omega_\ell^2}{v^2}\,\hat{U}_\ell \cdot \hat{U}_{\ell\pm1}\left(U_{\ell-1}\,e^{i\Phi} - U_{\ell+1}\,e^{-i\Phi}\right)$$

$$= -\frac{i}{2k\cos\theta}\left(\frac{\omega_\ell^2}{v^2} - k_\ell^2\right)U_\ell. \qquad (58)$$

To solve this equation, we follow the procedure of Klein and Cook[12] and write it in terms of three dimensionless parameters defined below

$$\zeta = \frac{\Delta n}{n}\,\frac{L}{k\cos\theta}\,\frac{\omega_\ell^2}{v^2}\,\hat{U}_\ell \cdot \hat{U}_{\ell\pm1}$$

$$\simeq \frac{\Delta n}{n}\,\frac{kL}{\cos\theta}\,\hat{U}_\ell \cdot \hat{U}_{\ell\pm1}, \qquad (59)$$

$$\alpha = \frac{k}{K}\sin\theta, \qquad (60)$$

$$Q = \frac{K^2L}{k\cos\theta}\,. \qquad (61)$$

In Eq. (59) we have assumed that $\omega_\ell \simeq \omega$ since $\Omega \ll \omega$. To express the right-hand side of Eq. (58) in terms of α and Q we note that

$$\frac{1}{k\cos\theta}\left(\frac{\omega_\ell^2}{v^2} - k_\ell^2\right) = \frac{1}{k\cos\theta}\left[K^2\ell^2\left(\frac{V^2}{v^2} - 1\right)\right.$$

$$\left. + 2k\ell K\left(\frac{V}{v} + \sin\theta\right)\right]$$

$$\simeq \frac{1}{k\cos\theta}\left(-K^2\ell^2 + 2k\ell K\sin\theta\right)$$

$$= \frac{K^2\ell}{k\cos\theta}\left(\frac{2k}{K}\sin\theta - \ell\right)$$

$$= \frac{Q}{L}\,\ell(2\alpha - \ell)$$

where we have neglected terms containing V/v and V^2/v^2 since $V/v << 1$. This is equivalent to assuming $\omega_\ell \simeq \omega$ so that $\omega_\ell^2/v^2 \simeq k^2$. Thus Eq. (58) in terms of ζ, α, and Q becomes

$$\frac{dU_\ell}{dz} + \frac{\zeta}{2L}\left(U_{\ell-1}\, e^{i\Phi} - U_{\ell+1}\, e^{-i\Phi}\right) = i\,\frac{Q}{2L}\,\ell(\ell-2\alpha)U_\ell. \tag{62}$$

If we let

$$U_\ell = V_\ell\, e^{i\ell\Phi}, \tag{63}$$

Eq. (63) can be written in terms of V_ℓ as

$$\frac{dV_\ell}{dz} + \frac{\zeta}{2L}\,(V_{\ell-1} - V_{\ell+1}) = i\,\frac{Q}{2L}\,\ell(\ell - 2\alpha)V_\ell. \tag{64}$$

This is a set of coupled difference-differential equations relating the amplitudes of the diffracted plane waves, derived by Raman and Nath in 1936.[5] The parameter ζ (called the Raman-Nath parameter) is, within a factor of $\hat{U}_\ell \cdot \hat{U}_{\ell\pm1}$, the magnitude of the phase modulation of the zero order light across the sound beam of width L. The variable α is a measure of the angle of incidence, and the parameter Q, as we shall see, describes the general nature of the diffraction process. Note that light-diffraction experiments with identical values of ζ, α, and Q are equivalent even though the values of λ_a (vacuum wavelength), n, K, Δn, or L may be different.

The phase factor

$$\frac{Q}{2L}\,\ell(\ell - 2\alpha), \qquad \textit{Phase Factor} \tag{65}$$

in Eq. (64) consists of two terms that may be interpreted as follows. When the light is incident normally so that $\alpha = 0$, only the first term remains. This term represents the phase difference between the ℓth-order and undiffracted (zero order) waves due to their different directions of propagation. Their phase difference after traversing the sound beam is given by

$$kL(\sec\theta_\ell - 1) \simeq \tfrac{1}{2}kL\,\tan^2\theta_\ell = \frac{Q\ell^2}{2L} \tag{66}$$

where in the last step we have substituted for θ_ℓ from Eq. (52) with $\theta = 0$. It is evident that the second term in the phase factor arises because of oblique incidence. To understand it physically, let us assume that Q is small so that the first term can be neglected. Then Eq. (60) becomes

$$\begin{aligned}\frac{dV_\ell}{dz} + \frac{\zeta}{2L}\left(V_{\ell-1} - V_{\ell+1}\right) &= -i\,\frac{Q}{L}\,\ell\alpha\, V_\ell \\ &= -iK\ell\,\tan\theta V_\ell\end{aligned} \tag{67}$$

where we have substituted for Q and α from Eqs. (60) and (61). If we let

$$V_\ell = W_\ell e^{iK\ell z \tan\theta}, \tag{68}$$

Eq. (67) can be written in terms of W_ℓ as

$$\frac{dW_\ell}{dz} + \frac{\zeta}{2L}\left(W_{\ell-1} e^{iKz \tan\theta} - W_{\ell+1} e^{-iKz \tan\theta}\right) = 0. \tag{69}$$

From this equation, we see that the effect of oblique incidence is to introduce a phase of $Kz \tan\theta$. This phase arises because at oblique incidence the light encounters a continually changing phase of the acoustic wave. When Q is large, both phase terms must be retained.

According to coupled mode (wave) theory, the parameter ζ is the coupling coefficient and couples directly only the adjacent modes. The normal modes are the partial waves in Eq. (48). The amount of energy transfer between any two waves depends on their degree of synchronization and the coupling coefficient. Two waves with the same phase factor are synchronous, and the energy transfers from one to the other most efficiently. When Q is small, the adjacent waves are nearly synchronous, and the energy transfers from the zero (which initially contains all the energy) to the first order, first to the second, second to the third, etc. This is the case of Raman-Nath diffraction. However, when Q is large, energy can transfer efficiently only to the first order provided $\alpha \simeq \frac{1}{2}$. This corresponds to Bragg diffraction. Below we consider three cases, namely, $Q \ll 1$, $Q \gg 1$, and $Q \simeq 1$.

$Q \ll 1$, The Region of Raman-Nath Diffraction

When $Q \ll 1$, the right-hand side of Eq. (64) is usually neglected. However, unless θ is approximately zero, the parameter α may be quite large for low acoustic frequencies. It may be mentioned that in practice the maximum value of ℓ is usually 10 or less. Thus neglecting the first term, which depends on ℓ^2, but retaining the second, which contains the angular dependence, Eq. (64) becomes

$$\frac{dV_\ell}{dz} + \frac{\zeta}{2L}\left(V_{\ell-1} - V_{\ell+1}\right) = -i \frac{Q}{L} \ell\alpha V_\ell. \tag{70}$$

Using the identity for Bessel functions of the first kind (ref. 7, p. 22),

$$2 \frac{dJ_\ell(z)}{dz} = J_{\ell-1}(z) - J_{\ell+1}(z), \tag{71}$$

the solution of Eq. (70) is found to be

$$V_\ell(z) = e^{i\ell[\pi - (Q\alpha/2L)z]} J_\ell \left[\frac{2\zeta}{Q\alpha} \sin\left(\frac{Q\alpha z}{2L}\right)\right]. \tag{72}$$

Note that the exact solution for $\ell = 0$ is given by

$$V_0(z) = J_0(\zeta z/L). \tag{73}$$

The solution in Eq. (72) satisfies the initial conditions since

$$J_\ell(0) = \delta_{\ell 0}. \tag{74}$$

At z = L, the intensity of the ℓth diffraction order is given by

$$\begin{aligned} I_\ell(z{=}L) &= |V_\ell(z{=}L)|^2 \\ &= J_\ell^2 \left[\zeta \frac{\sin(Q\alpha/2)}{Q\alpha/2}\right] \\ &= J_\ell^2 \left[\zeta \operatorname{sinc}\left(\frac{L}{\Lambda} \tan\theta\right)\right] \end{aligned} \tag{75}$$

where we have substituted for α and Q from Eqs. (60) and (61), respectively. For normal incidence, the intensity becomes

$$I_\ell(z{=}L,\ \theta{=}0) = J_\ell^2(\zeta). \tag{76}$$

For perpendicular polarization, this result is the same as Eq. (14) since

$$\zeta_\perp(\theta{=}0) = \frac{\Delta n}{n} kL, \tag{77}$$

which is identical with $\Delta\phi$. In fact, since $\Lambda >> \lambda$, $\cos(\lambda/\Lambda) \simeq 1$ and, therefore, $\zeta_\perp \simeq \zeta_{||}$. Hence, in the region of Raman-Nath diffraction, the diffraction efficiency for parallel polarization is practically equal to that for perpendicular polarization.

In addition to the conclusions of Section 2 with respect to the symmetry of the diffraction pattern and intensities of the diffraction orders, we note that the angular dependence of the arguments of the Bessel functions is such that the effect of oblique incidence is to alter the effective value of ζ by a factor of $\operatorname{sinc}(Q\alpha/2\pi)$. Maximum diffraction occurs for normal incidence, and there will be secondary maxima for various oblique incidences. Whenever $Q\alpha$ is an integral multiple of 2π (excluding zero), all diffraction effects disappear. Moreover, the light intensities of the positive and negative ℓth orders vanish simultaneously when $\zeta \operatorname{sinc}(Q\alpha/2\pi)$ is equal to any root of the ℓth-order Bessel function. (This follows from $J_{-\ell} = (-1)^\ell J_\ell$.) This can be made to occur by changing ζ or α. Since $\operatorname{sinc}(Q\alpha/2\pi)$ is symmetric in α, the intensity of a particular order as a function of α is symmetric about $\alpha = 0$ (normal incidence).

It is not difficult to understand physically why the diffraction effects are maximum for normal incidence and disappear when

$$Q\alpha = 2\pi m, \quad m = \pm 1, \pm 2, \ldots . \tag{78}$$

In Section 2 we saw that the incident wavefront emerges from the sound column modulated in phase. The amount of light diffracted increases as the magnitude of the modulation increases. From Fig. 20 it is evident that the modulation magnitude is maximum when the light is incident normally and decreases as the angle of incidence increases. It approaches zero when the angle of incidence becomes θ_1, where $\tan\theta_1 = \Lambda/L$, and consequently no amount of light is diffracted. As the angle of incidence is further increased, the modulation approaches a smaller maximum and then decreases to zero when $\tan\theta_2 = 2\Lambda/L$, and so on.

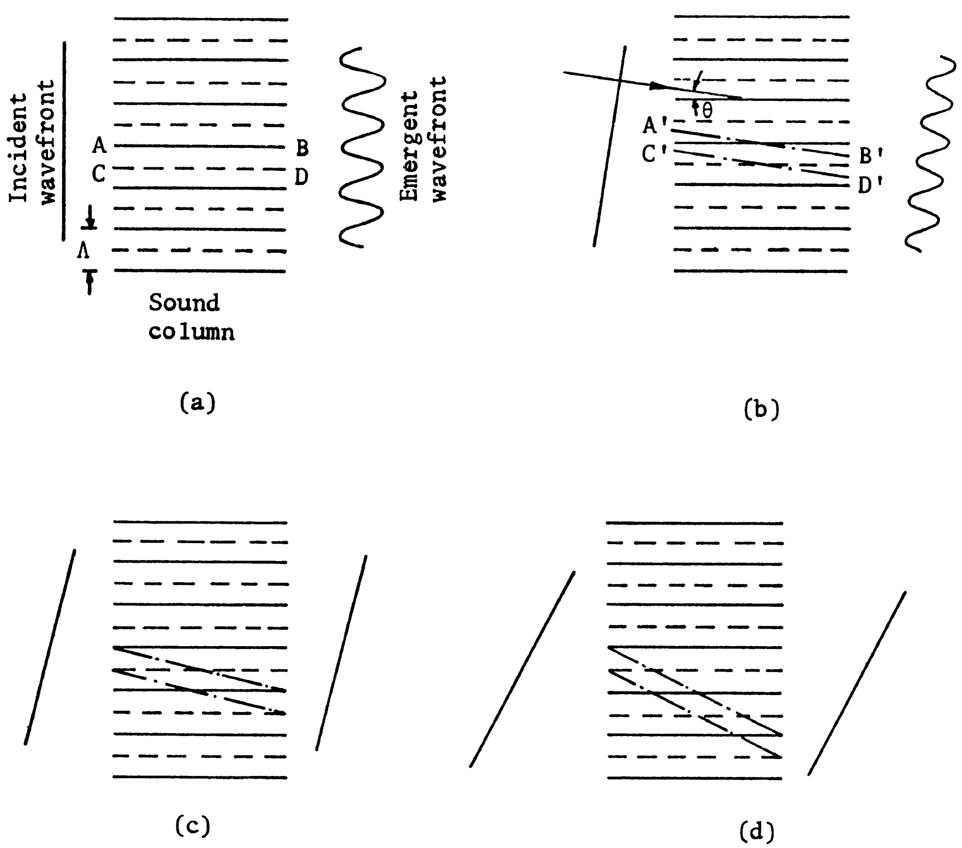


Fig. 20. Raman-Nath diffraction as a function of angle of incidence θ. *(a) Diffraction effects are maximum when* $\theta = 0$. *(b) Diffraction effects decrease as* θ *is increased since the amplitude of modulation decreases. Note that the optical path difference between paths* A'B' *and* C'D' *is less than that between* AB *and* CD *since the optical path along* A'B' *is less than that along* AB *and the optical path along* C'D' *is greater than that along* CD. *(c) and (d) Diffraction effects disappear when* $\tan\theta = m\Lambda/L$ *where* $m = \pm 1, \pm 2, \pm 3$, *etc. In (c) and (d),* $m = 1$ *and* 2, *respectively.*

Thus diffraction effects disappear for angles of incidence given by

$$\tan\theta_m = m\Lambda/L. \tag{79}$$

At these angles of incidence the light enters and leaves the sound column at points differing in acoustic phase by $2\pi m$. Smaller maxima are obtained at intermediate angles. Note that Eqs. (78) and (79) are identical. It may be added that for oblique incidence (see Fig. 20b), one can show that the argument of the Bessel function in Eq. (75) represents the magnitude of phase modulation for the perpendicular component. Thus we find that acoustic waves characterized by the condition $Q \ll 1$ are equivalent to moving optical gratings that produce only a modulation of the phase of light passing through them.

$Q \gg 1$, The Region of Bragg Diffraction

For large Q light can be transferred from the zero order to the ℓth order provided $\alpha = \ell/2$ (or nearly so), for then the phase factor vanishes for the ℓth order also, and the two waves are synchronous. However, the coupling coefficient ζ couples directly only the adjacent orders. Hence, the light is transferred efficiently to the positive first order only when $\alpha \simeq \frac{1}{2}$ and to the negative first order only when $\alpha \simeq -\frac{1}{2}$. For $\alpha \simeq \frac{1}{2}$, Eq. (64) yields the following two equations,

$$\frac{dV_1}{dz} - \frac{\zeta}{2L} V_1 = 0 \tag{80}$$

$$\frac{dV_0}{dz} + \frac{\zeta}{2L} V_0 = i\,\frac{Q}{2L}(1 - 2\alpha)V_1, \tag{81}$$

where the first equation corresponds to $\ell = 0$ and the second to $\ell = 1$. Solving these equations subject to the boundary conditions

$$\begin{aligned} V_0(z=0) &= 1 \\ V_1(z=0) &= 0 \end{aligned} \tag{82}$$

we obtain the following expressions for U_0 and U_1:

$$U_0(z) = e^{i(Q/2L)(\frac{1}{2}-\alpha)z}\left(\cos\left\{\frac{z}{2L}\left[Q^2(\tfrac{1}{2}-\alpha)^2 + \zeta^2\right]^{\frac{1}{2}}\right\} - iQ(\tfrac{1}{2}-\alpha)\,\frac{\sin\{(z/2L)[Q^2(\frac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}\}}{[Q^2(\frac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}}\right), \tag{83}$$

$$U_1(z) = \zeta\, e^{i[(Q/2L)(\frac{1}{2}-\alpha)z + \pi + \Phi]}$$

$$\cdot \frac{\sin\{(z/2L)[Q^2(\frac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}\}}{[Q^2(\frac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}} \quad . \tag{84}$$

The corresponding intensities are given by

$$I_0(z) = |U_0(z)|^2 = \cos^2\{(z/2L)[Q^2(\tfrac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}\}$$

$$+ \frac{Q^2(\frac{1}{2}-\alpha)^2}{Q^2(\frac{1}{2}-\alpha)^2 + \zeta^2} \sin^2\{(z/2L)[Q^2(\tfrac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}\}$$

$$= \frac{1}{Q^2(\frac{1}{2}-\alpha)^2 + \zeta^2} \Big(Q^2(\tfrac{1}{2}-\alpha)^2 + \zeta^2 \cos^2\{(z/2L)$$

$$\cdot [Q^2(\tfrac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}\}\Big), \tag{85}$$

$$I_1(z) = |U_1(z)|^2 = \frac{\zeta^2}{Q^2(\frac{1}{2}-\alpha)^2 + \zeta^2}$$

$$\cdot \sin^2\{(z/2L)[Q^2(\tfrac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}\}. \tag{86}$$

Note that $I_0 + I_1 = 1$, as it should under the assumption that all orders except $\ell = 0,1$ have zero intensity. Results similar to Eqs. (85) and (86) were obtained by Phariseau in 1956.[13]

At $z = L$, the amount of diffracted light is given by

$$I_1(z{=}L) = \frac{\zeta^2}{Q^2(\frac{1}{2}-\alpha)^2 + \zeta^2} \sin^2\{\tfrac{1}{2}[Q^2(\tfrac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}\}$$

$$= (\zeta/2)^2 \operatorname{sinc}^2\{(1/2\pi)[Q^2(\tfrac{1}{2}-\alpha)^2 + \zeta^2]^{\frac{1}{2}}\}. \tag{87}$$

The first zero of diffracted light as a function of α occurs when

$$|\alpha - \tfrac{1}{2}| = (1/Q)(4\pi^2 - \zeta^2)^{\frac{1}{2}}. \tag{88}$$

Thus the distribution of diffracted light as a function of α becomes sharper as we increase ζ or Q. Since ζ is usually small whereas Q (which depends on the square of the acoustic frequency) can vary over several orders of magnitude, the sharpness effect is more pronounced as a function of Q. In this sense, Q may be called the sharpness parameter of Bragg diffraction.

From Eq. (86) we note that if $0 < \zeta \leq 2\pi$, the maximum value of I_1 occurs when $\alpha = \frac{1}{2}$, i.e., when

$$\sin\theta = \frac{K}{2k}, \qquad \textit{Bragg Condition} \tag{89}$$

or the Bragg condition, angle of incidence equal to the Bragg angle Θ is satisfied. The intensities at z = L in this case are given by

$$I_0(z=L,\ \alpha=\tfrac{1}{2}) = \cos^2(\zeta/2)$$

and

$$I_1(z=L,\ \alpha=\tfrac{1}{2}) = \sin^2(\zeta/2), \tag{90}$$

which are the same as obtained from Eqs. (24) provided the incident light is polarized perpendicular to the plane of incidence. Thus when $\alpha = \tfrac{1}{2}$, the energy is exchanged sinusoidally between the zero- and first-order beams. Complete transfer occurs when $\zeta = \pi$ where

$$\zeta = \frac{\Delta n}{n} \frac{kL}{\cos\Theta} \hat{U}_0 \cdot \hat{U}_1 \tag{91}$$

from Eq. (59). (Note that in Bragg diffraction we do not need to assume $\hat{U}_\ell \cdot \hat{U}_{\ell-1} = \hat{U}_\ell \cdot \hat{U}_{\ell+1}$ because in this case we have only two waves, namely, the zero- and first-order waves.) In other words, complete transfer occurs when L is of the order of $n/\Delta n$ optical wavelengths. Since $\Delta n \simeq 10^{-5}$, the transfer process is extremely adiabatic as was assumed in neglecting the second derivatives of the x and y components of $\vec{U}_\ell$ with respect to z. This is also true in the case of Raman-Nath diffraction where, for normal incidence, the zero-order beam disappears when $\zeta \simeq 2.4$ radians. When θ is nonzero, it disappears for larger values of ζ. From Eq. (52) we find that $\theta_1 = \Theta$ when $\alpha = \tfrac{1}{2}$ so that the angle between the diffracted and undiffracted light is 2Θ. Thus the parameter ζ for perpendicular and parallel polarizations can be written as

$$\zeta_\perp = \frac{\Delta n}{n} \frac{kL}{\cos\Theta}, \qquad \text{Perpendicular Polarization} \tag{92}$$

$$\zeta_{||} = \frac{\Delta n}{n} kL \frac{\cos 2\Theta}{\cos\Theta}, \qquad \text{Parallel Polarization} \tag{93}$$

respectively. For small Bragg angles, which is generally the case, $\cos 2\Theta$ is very nearly equal to 1 and therefore $\zeta_\perp$ and $\zeta_{||}$ are nearly equal. Consequently, the parallel and perpendicular components are diffracted with equal efficiency.

When $\alpha \neq \tfrac{1}{2}$, the diffracted intensity can be written in the form

$$I_1(z=L) = (\zeta/2)^2 \,\mathrm{sinc}^2\{(1/2\pi)[(Q\Delta\alpha)^2 + \zeta^2]^{\frac{1}{2}}\} \tag{94}$$

where $\Delta\alpha$ is the deviation of α from its Bragg value of $\tfrac{1}{2}$. A deviation of $\Delta\alpha$ can be produced by a deviation of Δk in the optical wavenumber, or $\Delta\Lambda$ in the acoustic wavelength, or $\Delta\theta$ in the angle of incidence, where

$$\Delta\alpha = \frac{\Delta k}{K}\sin\Theta = \frac{k}{2\pi}\Delta\Lambda \sin\Theta = \frac{k}{K}\cos\Theta\Delta\theta. \tag{95}$$

Figure 21 shows how the diffracted intensity varies as a function of $Q\Delta\alpha$ for various values of ζ. As we know from Eq. (88) the curves are slightly sharper for larger values of ζ. Moreover, the sidelobe intensity increases with increasing ζ, especially when ζ approaches $3\pi/2$. Note that Eq. (94) reduces to Eq. (29) when ζ is small and $\cos\theta \simeq \cos\Theta$, showing that the intensity of the diffracted light measured as a function of the angle of incidence determines the angular distribution of the acoustic energy in the limit of low acoustic intensities only.

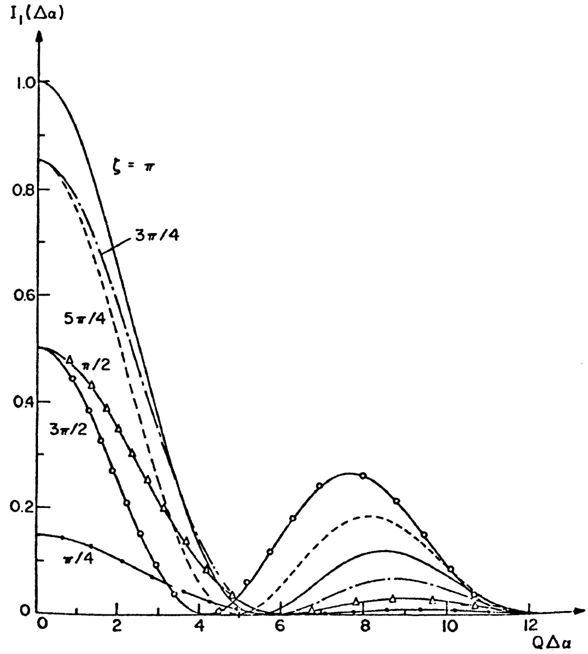


Fig. 21. Intensity of Bragg diffracted light as a function of deviation from the Bragg condition for various values of the Raman-Nath parameter.

Q ≃ 1, The Transition Region

Above we have found analytic solutions of the difference-differential equation (64) in the two limiting cases. For $Q \ll 1$, light is diffracted into several orders, which are symmetric about the incident light and whose amplitudes are given by Bessel functions of the first kind. For $Q \gg 1$, we have shown that it is diffracted into one of the first orders provided it is incident at an angle approximately equal to the corresponding Bragg angle. In this limit, the diffraction effects disappear when the deviation from the Bragg angle becomes large.

Outside these two limits, when $Q \simeq 1$, simple analytic solutions of Eq. (64) cannot, in general, be obtained. However, this equation can be solved numerically using a computer as has been done by Klein and Cook.[12] Their results clearly show how the transition from the Raman-Nath to Bragg diffraction takes place as a function of Q, and the curious reader should refer to their paper for details. Some of the points, however, are mentioned below.

For $\zeta \leq 3$, the diffraction effects almost disappear for normal incidence when Q is equal to 10 or larger. For oblique incidence the zero-order light is symmetric about $\alpha = 0$, i.e., $\theta = 0$, but the first orders are symmetric about $\alpha = \pm\frac{1}{2}$, i.e., $\theta = \pm\Theta$. In general, for $\zeta < 2\pi$, the regions of diffraction are divided as follows:

$$Q \leq 0.5 \qquad \textit{Raman-Nath Diffraction} \tag{96}$$

$$Q \geq 10 \qquad \textit{Bragg Diffraction} \tag{97}$$

The intermediate values of Q belong to the transition region. It is interesting to note that the condition $Q \geq 10$ implies a transition length of

$$L_t = 10 \frac{\Lambda^2}{2\pi\lambda} \cos\Theta,$$

which is approximately the same as in Eq. (15).

5. CONCLUSIONS

The diffraction of light by sound waves is described from three points of view: (1) constructive and destructive interference of the light scattered by sound waves, (2) diffraction in terms of photon-phonon interaction, and (3) diffraction according to the optical wave equation. Only the third gives the complete answer and incorporates the results of the first two. The difference between the solutions of the scalar and vector wave equations is the introduction of the cosine of the angle between two successive orders in the Raman-Nath parameter when the incident field component is polarized in the scattering plane. There is no difference for the perpendicular component.

In the Raman-Nath region of diffraction, i.e., narrow sound beams and low frequencies, the sound column behaves like a moving optical phase grating. The incident light is diffracted into many orders, and the frequency of the ℓth order is shifted from that of the incident light by ℓ times the acoustic frequency. The angle between two successive orders is small so that its cosine is very close to unity. Consequently, the Raman-Nath parameter is the same for both the parallel and perpendicular components. The two components are diffracted with equal efficiency unless the amplitudes of the index waves corresponding to them are different. Thus, in liquids, the two components are diffracted with equal efficiency. In solids, the amplitudes of the index waves corresponding to the parallel and perpendicular waves are generally different, causing the polarization of the diffracted waves to be different from each other and from that of the incident light.

In the Bragg region of diffraction, i.e., for wide sound beams and high frequencies, all diffraction effects disappear owing to destructive interference unless the light is incident at the Bragg angle, in which case it is diffracted into the order corresponding to a reflection of the incident light. The frequency of the Bragg diffracted light is upshifted or downshifted depending on the relative directions of the optical and acoustic k-vectors. For small Bragg angles, which is generally the case, the cosine of the angle between the diffracted and undiffracted beams is approximately unity, and the polarization of the two waves will be different from each other and from that of the incident wave only if the index waves corresponding to the parallel and perpendicular components have different amplitudes. For large Bragg angles, corresponding to higher acoustic frequencies (~10 GHz), the cosine factor becomes significant and causes the polarization of the waves to be different from each other even if the index waves for the two components are identical. Since the direction of propagation and phase of the diffracted wave depends upon the acoustic frequency and phase, respectively, Bragg diffraction provides a convenient way of controlling the direction of propagation and phase of the diffracted wave. Moreover, at low acoustic powers, since the intensity of the diffracted wave depends on the angular distribution of the acoustic energy, Bragg diffraction also provides a convenient way of studying the latter or in turn the properties of the medium and the transducer.

APPENDIX A. ACOUSTIC POWER AND THE CHANGE IN REFRACTIVE INDEX

When a solid is strained, its refractive index changes (photoelastic effect). The index change Δn is related to strain S through the photoelastic constant p of the material. This constant by definition is the change in reciprocal dielectric constant per unit strain.[14] Since the dielectric constant is equal to the square of the refractive index, the relation between Δn and S becomes

$$\Delta n = -\frac{n^3}{2} pS. \qquad (A.1)$$

In Eq. (A.1) p is the appropriate effective photoelastic constant giving the corresponding Δn. The negative sign indicates that when strain is positive, the index change is negative.

Whereas in solids the index change produced by a sound wave depends upon the direction of propagation and polarization of acoustic and optical waves, there can be no such dependence in the case of liquids.[15] Only longitudinal waves can propagate in liquids, and the index change produced by them can be obtained from the Lorentz-Lorenz relation. Thus

$$\Delta n = -\frac{(n^2-1)(n^2+2)}{6n} S. \qquad (A.2)$$

The index change produced by a unit strain is approximately 0.3 for most liquids.

The acoustic energy density (energy per unit volume) is given by $\frac{1}{2}CS^2$ or $\frac{1}{2}\rho V^2 S^2$ where C is the elastic stiffness constant and ρ is the mass density of the material. V as before is the velocity of sound in the material. The acoustic power density or intensity (i.e., power per unit area) is V times the energy density. In terms of the material constants, the acoustic power P_{ac} can be written as

$$P_{ac} = \tfrac{1}{2} HL\rho V^3 S^2 \qquad (A.3)$$

where H is the height (dimension perpendicular to the zx plane) and

L is the width of the sound beam. The quantity HL is the area of an acoustic wavefront and is approximately equal to the area of the transducer. From Eqs. (A.1), (A.2), and (A.3) we get a relation between Δn and P_{ac}:

$$\Delta n = -\frac{n^3}{2}\, p \left(\frac{2P_{ac}}{HL\rho V^3}\right)^{\frac{1}{2}} \qquad \textit{Solids} \qquad (A.4)$$

$$\Delta n = -\frac{(n^2-1)(n^2+2)}{6n} \left(\frac{2P_{ac}}{HL\rho V^3}\right)^{\frac{1}{2}}. \qquad \textit{Liquids} \qquad (A.5)$$

APPENDIX B. DOPPLER INTERPRETATION OF THE FREQUENCY SHIFTS

From the discussion above, we note that the frequency ω_ℓ of light in the ℓth diffraction order is shifted from the incident frequency ω (the frequency of the zero-order light) by $\ell\Omega$ where Ω is the frequency of the acoustic wave traveling with velocity V. This frequency shift can be interpreted as a Doppler shift due to the motion of the acoustic wavefronts. From Fig. B we see that light waves scattered from successive acoustic wavefronts interfere constructively when

$$\frac{2\pi}{\lambda}\Lambda \sin\theta + \frac{2\pi}{\lambda_\ell}\Lambda \sin\theta_\ell = 2\pi\ell, \qquad \text{(B.1)}$$

i.e., when their phase difference is an integral multiple of 2π. Equation (B.1) can be written in terms of the wavenumbers as

$$k \sin\theta + k_\ell \sin\theta_\ell = \ell K. \qquad \text{(B.2)}$$

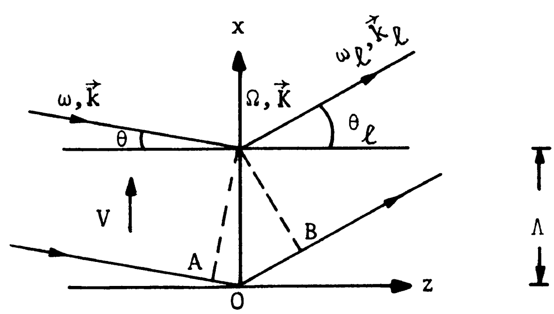


Fig. B. Scattering of light from moving acoustic wavefronts. Light scattered from successive acoustic wavefronts interferes constructively when it makes an angle θ_ℓ *with them where* OA = $\Lambda \sin\theta$, OB = $\Lambda \sin\theta_\ell$, *and* $k \cdot OA + k_\ell \cdot OB = 2\pi\ell$.

According to the special theory of relativity, if a light wave has a frequency ω and wavevector $\vec{k}$ in the lab frame, its frequency ω' in a frame moving with a velocity $\vec{V}$ with respect to the lab frame is given by

$$\omega' = \frac{1}{(1 - V^2/v^2)^{\frac{1}{2}}} (\omega - \vec{V} \cdot \vec{k}) \qquad (B.3)$$

where v = c/n is the speed of light in the medium. Equation (B.3) is the relativistic Doppler formula relating the frequency ω measured by an observer to the frequency ω' of a source moving with a velocity $\vec{V}$ with respect to the observer.

In the present problem, the acoustic wavefronts that generate the various orders of diffracted light act as a source moving with a velocity $\vec{V}$ parallel to the x axis. Applying Eq. (B.3) to both the diffracted and undiffracted (zero-order) light, we get

$$\omega' = \frac{1}{(1 - V^2/v^2)^{\frac{1}{2}}} (\omega + Vk \sin\theta) \qquad (B.4)$$

and

$$\omega_{\ell}' = \frac{1}{(1 - V^2/v^2)^{\frac{1}{2}}} (\omega_{\ell} - Vk \sin\theta_{\ell}), \qquad (B.5)$$

respectively. But in the frame of reference that is moving, the frequency of light in any diffraction order is the same as that of the incident light, i.e.,

$$\omega' = \omega_{\ell}' . \qquad (B.6)$$

This is also true in the lab frame in the case of a diffracting column that, instead of being space and time periodic, is space periodic only. A familiar example is that of a holographic record of a sinusoidal fringe pattern (hologram grating). From Eqs. (B.4), (B.5), and (B.6), we obtain

$$\omega_{\ell} = \omega + V(k \sin\theta + k_{\ell} \sin \theta_{\ell}). \qquad (B.7)$$

Or, using Eq. (B.2) and $\Omega = KV$, we get the desired result,

$$\omega_{\ell} = \omega + \ell\Omega. \qquad (B.8)$$

APPENDIX C. ACOUSTIC BEAM STEERING

Due to the diffraction spread of the acoustic beam (assuming that the incident light beam is a plane wave), the Bragg condition is satisfied over a finite range of the parameter α even though the direction of the acoustic beam is fixed. The range of α can be increased by using a narrow acoustic beam so that the acoustic energy is spread in a wide angular range. The scattering is evidently inefficient, but one could in principle supply the acoustic power needed to diffract a large fraction of the incident light. However, as the acoustic power is increased, its dissipation also increases. A large amount of power dissipation can set thermal gradients in the acoustic medium, which in turn can distort the light passing through the medium. The power requirements can be reduced if the transducer is designed to steer the radiated acoustic beam in such a way as to maintain the Bragg condition as α changes, i.e., if the acoustic wave direction can be changed as α changes so that the scattering is always at its optimum value.

To understand the theory of acoustic beam steering from a simple point of view, it is important to note that the angular distribution of acoustic power radiated from a transducer does not change (except for phase factors) as a function of distance from the transducer. Moreover, the amount of light diffracted at low acoustic powers (and consequently low diffraction efficiencies) does not depend on which part of the acoustic diffraction field the light traverses. Although the acousto-optic interaction usually takes place in the near acoustic field, analysis is convenient in the far field. Coquin et al.[16] have shown that the numerical results obtained by solving Eq. (62) for the case of ζ and Φ dependent on z (multi-element phased array transducer described below) are only slightly different from those one would obtain from this type of analysis even when as much as 80 percent of the incident light is diffracted.

Acoustic beam steering can be achieved if the single-element transducer is replaced by a multi-element phased array transducer shown in Fig. C. The transducer elements are equally spaced by a distance S and are driven separately by equal amplitude signals whose phases are advanced, from left to right, by an amount ψ per element. At a given time the wavefronts generated by the array form a staircase, as shown in Fig. C. This is equivalent in the far field of the array to

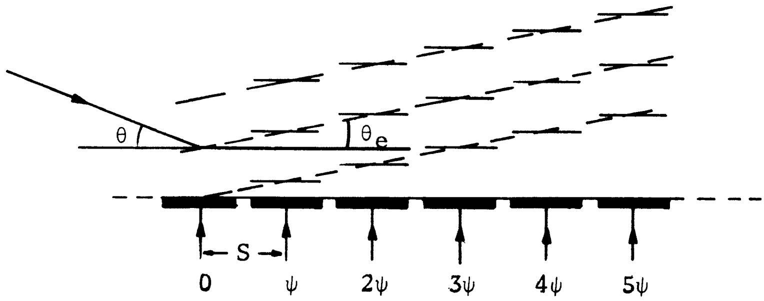


Fig. C. Multielement phased array transducer. Solid lines indicate actual acoustic wavefronts and dashed lines indicate the effective acoustic wavefronts. When light is incident making an angle θ with the plane of the array, the effective angle of incidence is $\theta + \theta_e$. *The angles shown are exaggerated for clarity.*

a planar wavefront making an angle θ_e with the transducer faces. Assuming that θ_e is small, one can write it in terms of ψ as

$$\theta_e \simeq \tan\theta_e = \psi/KS. \qquad \text{(C.1)}$$

If the incident light makes an angle θ with the plane of the array, the angle it makes with the effective acoustic planes is $\theta + \theta_e$. The incident light satisfies the Bragg condition if the angle $\theta + \theta_e$ is equal to the Bragg angle Θ. Accordingly we can define a beam steering error θ_{se} as the difference between these two angles, i.e.,

$$\theta_{se} = (\theta + \theta_e) - \Theta \qquad \textit{Steering Error} \qquad \text{(C.2)}$$

$$= \theta + \theta_e - \frac{K}{2k} \qquad \text{(C.3)}$$

where we have assumed $\Theta \simeq \sin\Theta = K/2k$.

Suppose that initially $\theta_{se} = 0$ so that the Bragg condition is satisfied. If now α changes, say, due to a change in the direction of the incident light, the direction of the diffracted light can be held fixed by changing the acoustic frequency by a proper amount. However, under these circumstances the Bragg condition is not satisfied. It can be satisfied if the acoustic beam is steered by choosing ψ properly so that the steering error is zero. Setting $\theta_{se} = 0$

in Eq. (C.3) and solving for the required perfect beam steering phase ψ_p, we obtain

$$\psi_p = KS\left(\frac{K}{2k} - \theta\right). \tag{C.4}$$

Note that ψ_p is a quadratic function of the acoustic wavenumber or frequency. Coquin et al.[16] have shown that a multi-element array steers the acoustic beam satisfactorily if the phase applied to each transducer is within $\pm\pi/4$ of the perfect beam steering phase at all frequencies. They find, for example, that with this type of steering the amount of light diffracted by a 10-element transducer is at the most 0.8 dB down from the case of perfect beam steering.

Some acoustic beam steering takes place even if ψ is fixed. The factor π is an experimentally convenient choice for a fixed ψ since it can be achieved by series interconnecting the elements.[11,17] Such a transducer is called a first-order beam-steered transducer. Since a planar phased array emits radiation in two equal lobes inclined in opposite directions with respect to the array plane and only one of them can be used for Bragg diffraction, half of the acoustic power is wasted. This power can be utilized if the elements are placed on steps (like a blazed grating) because the energy in that case is radiated in a single lobe.[16,18,19]

APPENDIX D. DIFFRACTION OF LIGHT BY STANDING SOUND WAVES

Consider a standing sound wave produced by the interference of two identical waves traveling in opposite directions parallel to the x axis. Such a standing wave is called a stationary wave. The corresponding index wave produced by it can be written as

$$\Delta n(x,t) = \Delta n[\sin(\Omega t - Kx) + \sin(\Omega t + Kx)], \tag{D.1}$$

or

$$\Delta n(x,t) = 2\Delta n \sin\Omega t \cos Kx. \tag{D.2}$$

Equation (D.2) represents a stationary index wave that is space periodic along the x axis with a spatial period $\Lambda = 2\pi/K$; its amplitude at a given point x is time dependent and given by $2\Delta n \sin\Omega t$. We assume that the sound column represented by Eq. (D.2) has a width L and a beam of light represented by

$$\vec{E}_i = \tfrac{1}{2}U_0 \, e^{i(\omega t - \vec{k}\cdot\vec{r})} + \text{c.c.} \tag{D.3}$$

is incident on it. In Eq. (D.3), the propagation vector $\vec{k}$ has the following components:

$$\vec{k} = (-k \sin\theta, 0, k \cos\theta). \tag{D.4}$$

We assume that the field at a point inside the sound column is given by

$$\vec{E} = \tfrac{1}{2} \sum \vec{U}_\ell(z,t) \, e^{i(\omega t - \vec{k}_\ell\cdot\vec{r})} + \text{c.c.} \tag{D.5}$$

where

$$\vec{k}_\ell = (-k \sin\theta + \ell K, 0, k \cos\theta). \tag{D.6}$$

The time-dependent amplitudes $\vec{U}_\ell(z,t)$ are obtained by substituting Eqs. (D.2) and (D.5) in the optical wave equation (46) and solving it. Proceeding as in Section 4, we obtain the following set of coupled difference-differential equations:

$$\frac{dU_\ell}{dz} + \frac{\zeta}{2L}\, 2\sin\Omega t \left(U_{\ell-1}\, e^{i\pi/2} - U_{\ell+1}\, e^{-i\pi/2}\right) = i\,\frac{Q}{2L}\,\ell(\ell - 2\alpha)U_\ell \tag{D.7}$$

where ζ, α, and Q have the same meanings as before (Eqs. (59) through (61)). Equation (D.7) is similar to Eq. (62); the former can be obtained from the latter if in the latter we replace ζ by $2\zeta \sin\Omega t$ and Φ by $\pi/2$. (Note that whereas in Eq. (62) U_ℓ is time independent, in Eq. (D.7) it is time dependent.) Consequently, we can immediately write down the solution of Eq. (D.7) knowing the solution of Eq. (62).

Raman-Nath Region

From Eqs. (62) and (64) and the discussion above, solution of Eq. (D.7) can be written as

$$U_\ell(z,t) = e^{i\ell[(3\pi/2) - (Q\alpha/2L)z]} \cdot J_\ell\left[\frac{2\zeta}{Q\alpha}\, 2\sin\Omega t\, \sin\left(\frac{Q\alpha}{2L}\, z\right)\right] . \tag{D.8}$$

Since the amplitude $U_\ell(z,t)$ is periodic in time, the ℓth-order wave is not a simple one but a superposition of a number of waves given by the Fourier analysis of U_ℓ or, equivalently, J_ℓ. From Graf's generalization of Neumann's addition theorem of Bessel functions (ref. 7, pp. 359-361), it can be shown that

$$J_\ell(2\eta \sin\Omega t) = e^{-i\ell\pi/2} \sum_{m=-\infty}^{\infty} J_{\ell+m}(\eta) J_m(\eta) e^{i(\ell+2m)\Omega t} . \tag{D.9}$$

Substituting Eq. (D.9) in Eq. (D.8) we can write the amplitude at $z = L$ as

$$U_\ell(L,t) = e^{i\ell[\pi - (Q\alpha/2)]} \sum_{m=-\infty}^{\infty} J_{\ell+m}(\eta) J_m(\eta) e^{i(\ell+2m)\Omega t} \tag{D.10}$$

where

$$\eta = \zeta\, \mathrm{sinc}\left(\frac{L}{\Lambda}\tan\theta\right). \tag{D.11}$$

From Eq. (D.10) it is clear that the ℓth-order wave is made up of several waves; its mth component has a frequency $\omega_{\ell m}$ and an intensity $I_{\ell m}$ given, respectively, by

$$\omega_{\ell m} = \omega + (\ell+2m)\Omega, \tag{D.12}$$

$$I_{\ell m} = J^2_{\ell+m}(\eta)J_m{}^2(\eta). \tag{D.13}$$

The direction of propagation of each component is the same and is given by Eq. (D.6). From Eq. (D.12) we note that all even orders (including the zero order) contain light of frequencies $\omega \pm 2r\Omega$ and all odd orders contain light of frequencies $\omega \pm (2r+1)\Omega$ where r is a positive integer (including zero). This is indicated in Fig. D.1. If we let $\ell = 2q$ and $\ell + 2m = r$, the intensity of the component with frequency $\omega \pm 2r\Omega$ in the 2qth order can be written as $J^2_{q-r}J^2_{q+r}$. Similarly, for odd orders, $\ell = 2q + 1$, the intensity of the component with frequency $\omega \pm (2r+1)\Omega$ in the (2q+1)th order can be written as $J^2_{q-r}J^2_{q+r+1}$.

The total instantaneous intensity in the ℓth diffraction order is given by

$$I_\ell(L,t) = |U_\ell(L,t)|^2 = J_\ell{}^2(2\eta \sin\Omega t)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} J_{\ell+m}(\eta)J_m(\eta)J_{\ell+p}(\eta)J_p(\eta)e^{i2(m-p)\Omega t}. \tag{D.14}$$

Its time average value is given by

$$\langle I_\ell \rangle = \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} I_\ell(t)dt \tag{D.15}$$

$$= \sum_{m=-\infty}^{\infty} J^2_{\ell+m}(\eta)J_m{}^2(\eta). \tag{D.16}$$

Bragg Region

Using Eqs. (83) and (84), the solution of Eq. (D.7) in the Bragg region, assuming that the Bragg condition ($\alpha = ½$) is satisfied, can be written as

$$U_0(L,t) = \cos(\zeta \sin\Omega t) \tag{D.17}$$

and

$$U_1(L,t) = -i \sin(\zeta \sin\Omega t) \tag{D.18}$$

where U_0 and U_1 are the amplitudes of the undiffracted (zero order) and diffracted (positive first order) waves. The diffracted as well as the undiffracted light contains waves of several frequencies as

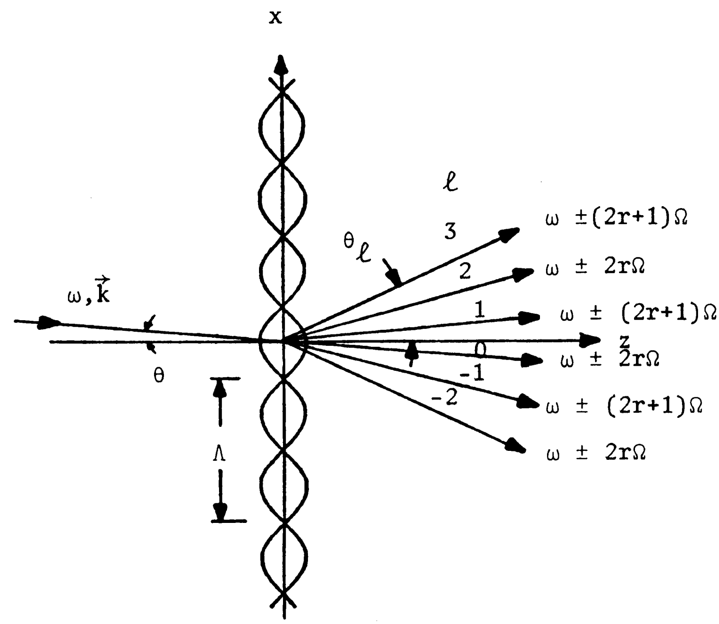


Fig. D.1. Raman-Nath diffraction of light by a standing sound wave. The light intensity of any diffraction order is time dependent. Fourier analysis of light shows that each even order (including the zero order) contains frequencies $\omega \pm 2r\Omega$, *and each odd order contains frequencies* $\omega \pm (2r+1)\Omega$, *where* $r = 0, 1, 2, \ldots.$ *The intensity of the* r*th component in the* 2q*th order is given by* $J^2_{q-r}J^2_{q+r}$. *Similarly, its intensity in the* (2q+1)*th order is given by* $J^2_{q-r}J^2_{q+r+1}$. *The total instantaneous intensity of the* ℓ*th order beam is given by* $J^2_{\ell}(2\eta \sin\Omega t)$ *where* $\eta = \zeta\ \mathrm{sinc}[(L/\Lambda)\tan\theta]$.

can be seen from the Fourier analysis of their amplitudes. The undiffracted amplitude U_0 can be expanded in a Fourier series as

$$U_0(L,t) = J_0(\zeta) + \sum_{m=1}^{\infty} J_{2m}(\zeta)\left[e^{i2m\Omega t} + e^{-i2m\Omega t}\right]. \quad (D.19)$$

It is evident that the undiffracted light is made up of waves of frequencies $\omega \pm 2m\Omega$ and corresponding intensities ${J_{2m}}^2$, where m is a positive integer (including zero). Similarly, writing the diffracted amplitude in a Fourier series,

$$U_1(L,t) = e^{i\pi} \sum_{m=0}^{\infty} J_{2m+1}(\zeta)\left[e^{i(2m+1)\Omega t} - e^{-i(2m+1)\Omega t}\right], \quad (D.20)$$

we note that the diffracted light contains waves of frequencies $\omega \pm (2m+1)\Omega$ and corresponding intensities J^2_{2m+1}, where m is a positive integer (including zero).

The instantaneous intensities of the undiffracted and diffracted light are given by

$$I_0 = \cos^2(\zeta \sin\Omega t) \tag{D.21}$$

and

$$I_1 = \sin^2(\zeta \sin\Omega t), \tag{D.22}$$

respectively. Their time average values are correspondingly given by

$$\langle I_0 \rangle = \tfrac{1}{2} + \tfrac{1}{2} J_0(2\zeta) \tag{D.23}$$

and

$$\langle I_1 \rangle = \tfrac{1}{2} - \tfrac{1}{2} J_0(2\zeta). \tag{D.24}$$

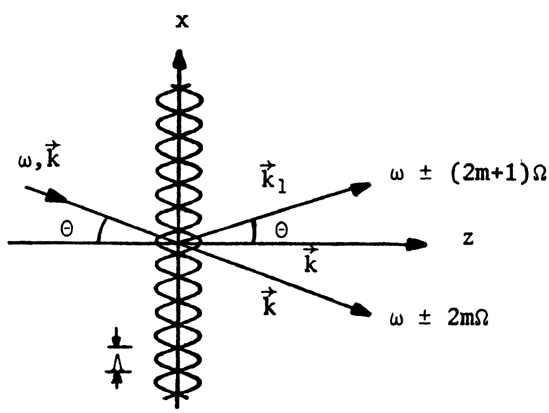


Fig. D.2. Bragg diffraction of light by a standing sound wave. The diffracted light has an intensity of $\sin^2(\zeta \sin\Omega t)$ *and contains frequencies* $\omega \pm (2m+1)\Omega$ *where* m = 0, 1, 2, ... *Undiffracted light has an intensity of* $\cos^2(\zeta \sin\Omega t)$ *and contains frequencies* $\omega \pm 2m\Omega$.

ACKNOWLEDGMENT

This work is a product of a suggestion by Dr. J. D. Gaskill in the summer of 1972 to look into wavefront correction through interaction of light with sound. The author gratefully acknowledges several discussions with him and with Dr. A. S. Marathay. He thanks Dr. Gaskill for reading the manuscript, Kathy Seeley and Janet Rowe for editing and typing, and Don Cowen for making the drawings.

The work was supported by the U.S. Army under contract DAAD07-72-C-0142.

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